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In the name of God

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(COAM)

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References

[1] "Standard Ethics", approved by Vice-Presidency for Research & Technology, the Ministry of Science, Research and Technology.

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Editor in Chief's Letter

It would be our great honor to have you as the readers of Journal of "Control and Optimization in Applied Mathematics (COAM)". The present journal is published and supported by Payame Noor University (PNU) as a semi-annual journal. Our main objective is to facilitate scientific regional and global discussions and collaborations between specialists in different fields of applied mathematics, especially in the fields of control and optimization. We hope that scholars and experts of different fields of applied mathematics find our scientific journal a platform for international communications of insight and knowledge. To assure the respectful subscribers about high quality of the journal, each article is reviewed by subject-qualified referees, the same as any other well-known international journal of applied mathematics.We believe that by publishing high quality and creative researches, we will observe more collaborations with our journal. We kindly invite all applied mathematicians especially in the fields of control and optimization, to join us by submitting their original works to the Journal of "Control and Optimization in Applied Mathematics". I want to thank the respectful colleagues of COAM, as well as referees, reviewers, and editors for their kind dedication and vision.

Contents

Control and Optimization in Applied Mathematics (COAM)
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Necessary Optimality Conditions for Non-smooth Continuous-Time Problems Using Convexificators

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Abstract. In this paper, we develop general necessary optimality conditions of the KKT types for non-smooth continuous-time optimization problems with inequality constraints. The primary instrument in our study is the concept of a convexificator. Based on this concept, non-smooth versions of the Mangasarian-Fromovitz constraint qualification are presented. Then, we derive optimality conditions for this problem under weak assumptions. Indeed, the constraint functions and the objective function that exist in this problem are not necessarily differentiable or convex.

Keywords. Continuous-time problems, Optimality conditions, Upper semi-regular convexificator, Non-smooth analysis.

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1 Introduction

In the years that have passed, several classes of vector and scalar Continuous-time programming problems have been the subject of countless researches. First, Bellman [5] studied this class of problems in a linear case. He formulated the dual problem of this class and provided the duality relations. Investigation of the optimality conditions of KKT type for nonlinear continuous-time problems was first studied by Hanson and Mon [14]. In a smooth case, further generalizations of the FJ and KKT type of the optimality conditions for continuous-time nonlinear problems are found in [1, 12, 14, 24]. On the other hand, the FJ and KKT optimality conditions for Lipschitz continuous-time programming problems were established by Brandao et al. in [6], for the non-smooth case. Later on, by using the Clarke sub-differentials, Nobakhtian et al. in [19, 20] provided the necessary optimality conditions for non-smooth continuous-time problems with vector-valued objective functions.

In recent years, the idea of convexificator which is seen as a generalization of the sub-differential has been studied by many researchers to extend and reinforce different results in non-smooth analysis and optimization. In particular, the well-known sub-differentials, such as Michel-Penot sub-differential, Clarke sub-differential, and Treiman sub-differential, are convexificators for the locally Lipschitz functions. Unlike, well-known sub-differentials which are compact and convex, the convexificators are not necessarily convex or compact, but are always closed sets. This concept was first defined by Demyanov [8] in the form of a convex and compact set, as a generalization of the idea of lower concave and upper convex approximations. Afterward, Jeyakumar and Luc in [16] used a closed and nonconvex set instead of a convex and compact set and they introduced what is now known as the nonconvex convexifiector.

Many researchers have obtained different results concerning FJ and KKT type necessary optimality conditions for non-smooth optimization problems with using convexificators (see [3, 4, 8, 9, 10, 11, 13, 16, 22, 23]). The upper semi-regular convexificator defined in [11] is a strengthened version of an upper convexifactor. Further, for the locally Lipschitz function, one can have the upper semi-regular convexificators smaller than the Michel-Penot sub-differential, Clarke sub-differential, Mordukhovich sub-differential, and Treiman sub-differential, so the optimality conditions obtained in terms of upper semi-regular convexificators are sharper. Our purpose in this paper is to introduce a new generalization of the Mangasarian-Fromovitz constraint qualification and following that to obtain an actual result of necessary optimality conditions in the form of KKT type for non-smooth continuous-time problems with inequality constraints with using convexificator. As mentioned above, the convexificator is viewed as a generalization of the sub-differential, so it leads to more robust results in optimiza-

tion. Indeed, the results in this perusal are correct with the convexificator replaced by known sub-differentials such as Clarke and Mordukhovich sub-differentials where the functions are locally Lipschitz. The main result of this paper might be used to generalize the duality and optimality conditions for non-smooth continuous-time optimization problems in future research.

The structure of the next sections of this work is as follows. In the second section, we introduce the notations and give the basic definitions of convexificators, support functions, integration of Multifunctions, and derive some preparative results from being used in the residual of this paper. In the third section, we introduce a new constraint qualification for non-smooth continuous-time problems with inequality constraints via convexificator and we establish KKT type necessary optimality conditions.

2 Notations and Preliminaries

Let $\mathcal{Z} \subset L_m^\infty[0,\mathcal{T}]$ be a nonempty, open and convex where $(L_m^\infty[0,\mathcal{T}], \|.\|_\infty)$ is the Banach space of all m-dimensional vector-valued Lebesgue measurable essentially bounded functions defined on the $[0,\mathcal{T}] \subset \mathbb{R}$, with the essential norm $\|.\|_\infty$ appointed by

$$\|z\|_\infty = \max_{1\le j\le m} \operatorname{ess\,sup}\{|z_j(t)|, \quad t \in [0,\mathcal{T}]\},$$

for $z(t) = (z_1(t), \cdots, z_m(t)) \in \mathbb{R}^m$. We consider the following non-smooth continuous-time problem:

$$\begin{aligned} (CTP) \qquad \min \quad & \Theta(\mathcal{R}) = \int_0^{\mathcal{T}} \theta(t,\mathcal{R}(t))dt, \\ s.t. \quad & \rho_i(t,\mathcal{R}(t)) \le 0, \quad i \in \mathcal{I} = \{1,\dots,r\}, \quad a.e. \quad t \in [0,\mathcal{T}], \\ & \mathcal{R} \in \mathcal{Z}, \end{aligned}$$

where the functions $t \to \theta(t,\mathcal{R}(t))$ and $t \to \rho_i(t,\mathcal{R}(t))$ are integrable and Lebesgue measurable for all $\mathcal{R} \in \mathcal{Z}$. We define $\rho_i(t,\mathcal{R}(t)) = \Phi_i(\mathcal{R})(t)$, $i \in \mathcal{I}$ and $\theta(t,\mathcal{R}(t)) = \Psi(\mathcal{R})(t)$, where $\Phi_i(\cdot)$ and $\Psi(\cdot)$ are maps from $\mathcal{Z}$ into the normed space $\Lambda_1^1[0,\mathcal{T}]$ of all functions which are Lebesgue measurable and essentially bounded defined on the compact interval $[0,\mathcal{T}]$. In the rest of this paper, the set of feasible solutions of the (CTP) is denoted as Ω. In other words,

$$\Omega = \{\mathcal{R} \in \mathcal{Z} : \ \rho_i(t,\mathcal{R}(t)) \le 0 \quad a.e.\ t \in [0,\mathcal{T}], \quad i \in \mathcal{I}\}.$$

Next, we reminisce some concepts and substantial structures of non-smooth analysis that are needed in the following of this paper. Most of the concepts and substance

included here can be found in [7] where the reader can find rather references, details and discussions.

Suppose that $\psi : Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be an extended real valued function defined on the Banach space Y. The upper and lower Dini directional derivatives of ψ at $\bar{y}$ in direction $d \in Y$ are introduced respectively by

$$\psi^{+}(\bar{y};d) := \limsup_{t\downarrow 0} \frac{\psi(\bar{y}+td)-\psi(\bar{y})}{t},$$
$$\psi^{-}(\bar{y};d) := \liminf_{t\downarrow 0} \frac{\psi(\bar{y}+td)-\psi(\bar{y})}{t}.$$

Whenever for function ψ at $\bar{y}$ in direction $d \in Y$, the upper and lower Dini directional derivatives are equal, we say that the function ψ has directional derivative at $\bar{y}$ in direction $d \in Y$ and is denoted as $\psi'(\bar{y};d)$. If ψ is Fréchet differentiable at $\bar{y}$ with Fréchet derivative $\nabla\psi(\bar{y})$, then for all $d \in \mathbb{R}^m$, $\psi'(\bar{y};d) = \langle \nabla\psi(\bar{y}), d\rangle$. Also, it is worth mentioning that both the upper and lower Dini derivatives definitely exist for locally Lipschitz functions.

Now, assume $\overline{\mathcal{R}} \in \mathcal{Z}$ and $d \in L_m^{\infty}[0,\mathcal{T}]$. We have

$$\rho_i^{+}(t,\overline{\mathcal{R}}(t);d(t)) := \Phi_i^{+}(\overline{\mathcal{R}};d)(t) := \limsup_{\lambda\downarrow 0} \frac{\Phi_i(\overline{\mathcal{R}}+\lambda d)(t)-\Phi_i(\overline{\mathcal{R}})(t)}{\lambda},$$

a.e. $t \in [0,T]$. According to the above hypotheses, it follows that for every $\overline{\mathcal{R}} \in \mathcal{Z}$ and $d(t) \in L_m^{\infty}[0,\mathcal{T}]$ the functions

$$t \to \psi^{+}(t,\overline{\mathcal{R}}(t);d(t))$$
$$t \to \rho_i^{+}(t,\overline{\mathcal{R}}(t);d(t)), \quad i \in \mathcal{I},$$

are integrable and Lebesgue measurable.

Now, to achieve the desired results, we recall the definition of the integration of multi-functions. Let $\Gamma : [0,\mathcal{T}] \to \mathbb{R}^m$ be a multi-function defined on $[0,\mathcal{T}]$. The integral of Γ, which denoted by $\int_0^{\mathcal{T}} \Gamma(t)dt$, is defined by the following subset of $\mathbb{R}^m$,

$$\int_0^{\mathcal{T}} \Gamma(t)dt := \left\{ \int_0^{\mathcal{T}} \gamma(t)dt : \quad \gamma \in \mathcal{S}(\Gamma) \right\},$$

where $\mathcal{S}(\Gamma)$ is the following set,

$$\mathcal{S}(\Gamma) := \{\gamma \in L_1^m[0,\mathcal{T}],\ \gamma(t) \in \Gamma(t) \text{ a.e. } t \in [0,\mathcal{T}]\},$$

where $L_1[0,\mathcal{T}] := \{\eta : [0,\mathcal{T}] \to \mathbb{R} : \|\eta\|_1 := \int_0^{\mathcal{T}} |\eta(t)| < \infty\}$.

If for measurable multi-function Γ, there exists an integrable function $\beta : [0,\mathcal{T}] \to \mathbb{R}_+$ such that

$$|\Gamma(t)| = \sup_{\xi\in\Gamma(t)} \|\xi\| \leq \beta(t), \qquad a.e.\ t \in [0,\mathcal{T}],$$

thus, we say that Γ is integrally bounded.

Suppose that for Banach space Y, the dual space of continuous linear functionals on Y equipped with the weak* topology is denoted by Y^*. In this case, the support function of a nonempty subset Δ in Y is the function $H_\Delta : Y^* \to \overline{\mathbb{R}}$, which defined as follows

$$H_\Delta(\xi) = \sup\{\langle \xi, y\rangle : y \in \Delta\},$$

where $\langle\cdot,\cdot\rangle$ is the standard duality pairing between Y^* and Y.

Next, we state some fundamental results for support functions that are required in the sequel.

Proposition 1. ([15]) Let $\mu,\ \lambda \geq 0$ be given scalars. Then, for nonempty closed convex subsets $\Sigma,\ \Delta$ of Y, we have

$$\Sigma \subseteq \Delta \quad \text{iff} \quad H_\Sigma(\xi) \leq H_\Delta(\xi), \quad \forall \xi \in Y^*,$$
$$\mu H_\Sigma(\xi) + \lambda H_\Delta(\xi) = H_{\{\mu\Sigma+\lambda\Delta\}}(\xi), \quad \forall \xi \in Y^*.$$

Theorem 1. ([2, Proposition 8.6.2]) For an integrably bounded multi-function Γ with compact values, we have

$$H_{\int_0^T \Gamma(t)dt}(v) = \int_0^T H_{\Gamma(t)}(v)dt, \quad \forall\ v \in \mathbb{R}^n.$$

We now recall the definitions of convexificator and some of its significant properties from [16]. As mentioned before, this notion plays a key role in the main results of this paper.

- We say that the function $\psi : Y \to \overline{\mathbb{R}}$ have an upper convexificator (u.c or Jeyakumar-Luc sub-differential [23]) (or lower convexificator(l.c)) at $y \in Y$ if there is a weak* closed set $\partial^*\psi(y) \subset Y^*$ (or $\partial_*\psi(y) \subset Y^*$) so that for every $d \in Y$,

  $$\psi^-(y;d) \leq \sup_{\xi\in\partial^*\psi(y)} \langle \xi, d\rangle \Big(\text{or } \psi^+(y;d) \geq \inf_{\xi\in\partial_*\psi(y)} \langle \xi, d\rangle\Big).$$

  A closed set $\partial^*\psi(y) \subset Y^*$ is said to be a convexificator of ψ at y iff it is both l.c and u.c of ψ at y.

- The function $\psi : Y \to \overline{\mathbb{R}}$ is said to have an upper-regular convexificator(u.r.c)(or lower regular convexificator(l.r.c)) at $y \in Y$ if there is a weak* closed set $\partial^*\psi(y) \subset Y^*$ (or $\partial_*\psi(y) \subset Y^*$) so that for every $d \in Y$,

  $$\psi^+(y;d) = \sup_{\xi\in\partial^*\psi(y)} \langle \xi, d\rangle \Big(\text{or } \psi^-(y;d) = \inf_{\xi\in\partial_*\psi(y)} \langle \xi, d\rangle\Big).$$

Some significant properties of convexificators, such as neither necessarily convex nor necessarily compact, make it possible to study a large class of non-smooth problems using this concept. We note that if a continuous function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ embrace a locally bounded Jeyakumar-Luc sub-differential at z, then it is locally Lipschitz near this point (see [16]).

Now, we recall from [10], the definition of upper semi-regular convexificators which will be beneficial in what follows.

- We say that, the function $\psi : Y \to \overline{\mathbb{R}}$ have an upper-semi regular convexificator(u.s.r.c) at $y \in Y$ if there is a closed set $\partial^*\psi(y) \subset Y^*$ so that for every $d \in Y$,

$$\psi^+(y;d) \leq \sup_{\xi \in \partial^*\psi(y)} \langle \xi, d \rangle.$$

Obviously, an u.r.c of ψ is also an u.s.r.c of ψ and each u.s.r.c is an u.p. Moreover, convex hull of an u.s.r.c of a locally Lipschitz function may be strictly contained in Michel-Penot and Clarke sub-differential(see Example 2.1 of [16]).

Example 1. Assume that function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as follows:

$$\psi(z) := \begin{cases} \sin 2z, & \text{if } \ z \in [0, +\infty) \cap \mathbb{Q}, \\ z^3 - 3z, & \text{if } \ z \in (-\infty, 0] \cap \mathbb{Q}, \\ 0, & \text{Otherwise}, \end{cases}$$

where $\mathbb{Q}$ is the set of rational numbers. In this case, by slightly calculating, the lower and upper Dini derivatives of function ψ at the point $z = 0$, are obtained respectively by:

$$\psi^-(0;d) = 0, \quad (\forall d \in \mathbb{R}),$$
$$\psi^+(0;d) = \begin{cases} 2d, & \text{if } \ d \geq 0, \\ -3d, & \text{if } \ d < 0. \end{cases}$$

Therefore, for function ψ at $z = 0$, the nonconvex set $\{-3, 2\}$ is an u.s.r.c. Also, the convex set $[-3, 2]$ is too another u.s.r.c for this function and so they are u.c for ψ at $z = 0$. Indeed, convexificators are not necessarily unique.

Since for all $d \in Y$, $\psi^-(y;d) \leq \psi^+(y;d)$, it follows that an u.s.r.c is also an u.c of ψ at y. The following example illustrates that the reverse of this statement is not necessarily true.

Example 2. Suppose that the propound function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is as follows.

$$\psi(z) = \begin{cases} \tan 2z, & \text{if } \ z \in \mathbb{Q}, \\ 0, & \text{if } \ z \notin \mathbb{Q}. \end{cases}$$

Then,

$$\psi^+(0;d) = 2\max\{0,d\},$$
$$\psi^-(0;d) = 2\min\{0,d\}.$$

Observe that the set $\partial^*\psi(0) = \{-1,1\}$ is an u.c of ψ at $z = 0$. On the other hand, since by considering $d = 1$, we have,

$$\sup_{\xi\in\partial^*\psi(0)} \langle \xi, 1\rangle = 1 < \psi^+(0;1) = 2.$$

The set $\partial^*\psi(0)$ is not an u.s.r.c.

Remark 1. Note that the set $\partial^*\psi(x) = \{\nabla\psi(x)\}$ is a unique u.r.c for any differentiable function such as ψ. Moreover, we know that any locally Lipschitz function is differentiable almost everywhere. Thus, these functions have a u.r.c on a dense set. Also, if ψ is locally Lipschitz, then the Mordukhovich sub-differential $\partial_M\psi(x)$ [18], Michel-Penot sub-differential $\partial^\diamond\psi(x)$ [17], Clarke sub-differential $\partial_C\psi(x)$ [7] and Treiman sub-differential $\partial_T\psi(x)$ [21] are examples of u.s.r.c for ψ.

In the following Lemma, we derive some calculus rules for upper semi-regular convexificator under convenient conditions.

Lemma 1. Let $\psi = (\psi_1,\dots,\psi_k)$ be a continuous vector function from Banach space Y to $\mathbb{R}^k$, and for each $l = 1,\dots,k$, the function ψ_l admits an u.s.r.c, $\partial^*\psi_l(\tilde{y})$ at $\tilde{y}\in Y$. Also, assume that,

$$g(y) = \max\{\psi_l(y) :\ l = 1,\dots,k\},$$

and $I(\tilde{y}) = \{l :\ g(\tilde{y}) = \psi_l(\tilde{y})\}$. Then $co\{\bigcup_{l\in I(\tilde{y})}\partial^*\psi_l(\tilde{y})\}$ is an u.s.r.c of g at $\tilde{y}$(for the nonempty set S, the convex hall of S is denoted by coS).

Proof. Suppose, without loss of generality, that $I(\tilde{y}) = \{1,\dots,q\}$. Thus

$$g(\tilde{y}) = \psi_1(\tilde{y}) = \cdots = \psi_q(\tilde{y}) > \psi_j(\tilde{y}),\quad \forall j = q+1,\dots,k.$$

From the continuity of ψ_l, for every y in a neighborhood of $\tilde{y}$, we have

$$g(y) = \max\{\psi_l(y) : l = 1,\dots,q\}.$$

Therefore

$$\begin{aligned} g^+(\tilde{y};d) &= \limsup_{t\downarrow o}\frac{g(\tilde{y}+td)-g(\tilde{y})}{t} \\ &= \limsup_{t\downarrow o}\frac{\max\{\psi_1(\tilde{y}+td),\dots,\psi_q(\tilde{y}+td)\}-g(\tilde{y})}{t} \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{t\downarrow o} \max\Big\{\frac{\psi_1(\tilde{y}+td)-\psi_1(\tilde{y})}{t},\ldots,\frac{\psi_q(\tilde{y}+td)-\psi_q(\tilde{y})}{t}\Big\}\\
&= \max\Big\{\limsup_{t\downarrow o}\frac{\psi_1(\tilde{y}+td)-\psi_1(\tilde{y})}{t},\ldots,\limsup_{t\downarrow o}\frac{\psi_q(\tilde{y}+td)-\psi_q(\tilde{y})}{t}\Big\}\\
&= \max\Big\{\psi_1^+(\tilde{y};d),\ldots,\psi_q^+(\tilde{y};d)\Big\}\\
&\leq \sup\Big\{\langle\eta,d\rangle : \eta\in\partial^*\psi_1(\tilde{y})\cup\ldots\cup\partial^*\psi_q(\tilde{y})\Big\}\\
&\leq \sup\Big\{\langle\eta,d\rangle : \eta\in co\{\bigcup_{l=1}^{q}\partial^*\psi_l(\tilde{y})\}\Big\}
\end{aligned}
$$

Hence, $co\{\bigcup_{l\in I(\tilde{y})}\partial^*\psi_l(\tilde{y})\}$ is an u.s.r.c of g at $\tilde{y}$. □

3 Main Results

In this section, we focus on getting optimality conditions for (CTP). Indeed, we deduce the KKT type of the necessary optimality conditions for a large class of nonconvex and non-differentiable continuous-time problems. Using the idea of u.s.r.c, we generalize the Mangasarian-Fromovitz constraint qualification for a non-smooth problem where the objective and constraint functions are continuous. The Generalized Gordan Theorem [25], will help to obtain the results of this section.

Let us introduce the following non-smooth analogue of the generalized Mangasarian–Fromovitz constraint qualification named by (GMFCQ).

Definition 1. The GMFCQ is satisfied at a feasible point $\tilde{x}$ of (CTP), if there exists a non-zero vector $v\in L^\infty[0,\mathcal{T}]$ such that for all $i\in\mathcal{I}$,

$$\sup_{\zeta\in\partial^*\rho_i(t,\tilde{x}(t))}\langle\zeta,v(t)\rangle<0,\qquad a.e.\ t\in A_i(\tilde{x}),$$

where $A_i(\tilde{x}):=\{t\in[0,\mathcal{T}]:\rho_i(t,\tilde{x}(t))=0\}$.

Next, we are ready to demonstrate our result of KKT type necessary conditions in terms of u.s.r.c.

Theorem 2. Let GMFCQ holds at $\bar{\mathcal{R}}\in\Omega$. Also, suppose that $\theta(t,\cdot)$ and $\rho_i(t,\cdot)$, $i\in\mathcal{I}$ are continuos functions at $\bar{\mathcal{R}}$, and admit bounded upper semi-regular convexificator $\partial^*\theta(t,\bar{\mathcal{R}})$ and $\partial^*\rho_i(t,\bar{\mathcal{R}})$ for all $i\in\mathcal{I}$. Then there exists $\overline{\lambda}\in L_r^\infty[0,\mathcal{T}]$, such that

$$0\in\int_0^{\mathcal{T}}\Big[co\partial^*\theta(t,\bar{\mathcal{R}}(t))+\sum_{i=1}^{r}\overline{\lambda}_i(t)co\partial^*\rho_i(t,\bar{\mathcal{R}}(t))\Big]dt, \tag{1}$$

$$\overline{\lambda}(t) \geq 0, \quad a.e.\ t \in [0,\mathcal{T}], \tag{2}$$
$$\overline{\lambda}_i(t)\rho_i(t,\bar{\mathcal{R}}(t)) = 0, \quad a.e.\ t \in [0,\mathcal{T}], \quad i \in \mathcal{I}. \tag{3}$$

Proof. We get,

$$\rho(t,\bar{\mathcal{R}}(t)) = \max_{1\leq i\leq r} \rho_i(t,\bar{\mathcal{R}}(t)), \quad a.e.\ t \in [0,\mathcal{T}].$$

Thus, we reformulate (CTP) in the following tantamount form.

$$\begin{aligned} \min \quad & \Theta(\mathcal{R}) = \int_0^{\mathcal{T}} \theta(t,\mathcal{R}(t))dt, \\ s.t. \quad & \rho(t,\mathcal{R}(t)) \leq 0, \quad a.e. \ \ t \in [0,\mathcal{T}], \\ & \mathcal{R} \in \mathcal{Z}. \end{aligned}$$

Now, we show that the following system has no solution $d \in L_r^\infty[0,\mathcal{T}]$.

$$\Theta^+(\bar{\mathcal{R}};d) < 0, \tag{4}$$
$$\rho^+(t,\bar{\mathcal{R}}(t);d(t)) < 0, \quad a.e.\ t \in A(\bar{\mathcal{R}}) := \{t \in [0,\mathcal{T}] : \rho(t,\bar{\mathcal{R}}(t)) = 0\}. \tag{5}$$

Assume, to the contrary, that the system (4) and (5) has a solution $\hat{d} \in L_r^\infty[0,\mathcal{T}]$. According to continuity of the functions and the limsup properties , there exists a real number $\delta > 0$ such that for every $\gamma \in (0,\delta)$,

$$\begin{aligned} &\rho(t,\bar{\mathcal{R}}(t) + \gamma\hat{d}(t)) \leq 0, \quad a.e.\ t \in [0,\mathcal{T}], \\ &\Theta(\bar{\mathcal{R}} + \gamma\hat{d}(t)) < \Theta(\bar{\mathcal{R}}), \quad \bar{\mathcal{R}} + \gamma\hat{d}(t) \in \mathcal{Z}. \end{aligned}$$

This means that $\bar{\mathcal{R}} + \gamma\hat{d}$ is a feasible solution of (CTP) for $\gamma \in (0,\delta)$. This conflicts with the fact that $\hat{z}$ is an optimal solution for problem (CTP). Hence there does not exists $d \in L_r^\infty[0,\mathcal{T}]$ which satisfy in the systems (4) and (5). Applying the Generalized Gordan Theorem yields the existence of $\lambda_0 \geq 0$, $\hat{\lambda}(t) \geq 0$ a.e. $t \in [0,\mathcal{T}]$, not all identically zero, such that

$$0 \leq \lambda_0\Theta^+(\bar{\mathcal{R}};d) + \int_{A(\bar{\mathcal{R}})} \hat{\lambda}(t)\rho^+(t,\bar{\mathcal{R}}(t);d(t))dt, \quad \forall d \in L_r^\infty[0,\mathcal{T}].$$

Setting $\lambda(t) = \hat{\lambda}(t)$, if $t \in A(\bar{\mathcal{R}})$ and $\lambda(t) = 0$, otherwise, we acquire

$$\begin{aligned} 0 &\leq \lambda_0\Theta^+(\bar{\mathcal{R}};d) + \int_0^{\mathcal{T}} \lambda(t)\rho^+(t,\bar{\mathcal{R}}(t);d(t))dt \\ &= \lambda_0 \limsup_{\mu\downarrow 0} \frac{\int_0^{\mathcal{T}} [\Psi(\bar{\mathcal{R}} + \mu d)(t) - \Psi(\bar{\mathcal{R}})(t)]dt}{\mu} + \int_0^{\mathcal{T}} \lambda(t)\rho^+(t,\bar{\mathcal{R}}(t);d(t))dt \\ &\leq \int_0^{\mathcal{T}} \lambda_0\Big[\limsup_{\mu\downarrow 0} \frac{\Psi(\bar{\mathcal{R}} + \mu d)(t) - \Psi(\bar{\mathcal{R}})(t)}{\mu}\Big]dt + \int_0^{\mathcal{T}} \lambda(t)\rho^+(t,\bar{\mathcal{R}}(t);d(t))dt \end{aligned}$$

$$= \int_0^{\mathcal{T}} \lambda_0 \theta^+(t,\bar{\mathcal{R}}(t);d(t))dt + \int_0^{\mathcal{T}} \lambda(t)\rho^+(t,\bar{\mathcal{R}}(t);d(t))dt,$$

for all $d \in L_r^\infty[0,\mathcal{T}]$. Let $I(t,\bar{\mathcal{R}}(t)) = \{i \in \mathcal{I} : \rho(t,\bar{\mathcal{R}}) = \rho_i(t,\bar{\mathcal{R}})\}$. Similar to the proof of Lemma 1, we have

$$\rho^+(t,\bar{\mathcal{R}}(t);d(t)) = \max\{\rho_i^+(t,\bar{\mathcal{R}}(t);d(t)) : i \in I(t,\bar{\mathcal{R}}(t))\}. \tag{6}$$

Now, let $d_0 \in L_r^\infty[0,\mathcal{T}]$ be the vector which is satisfied in $(GMFCQ)$. Therefore for every $i \in \mathcal{I}$,

$$\sup_{\zeta \in \partial^* \rho_i(t,\bar{\mathcal{R}}(t))} \langle \zeta, d_0(t)\rangle < 0, \qquad a.e.\ t \in A_i(\bar{\mathcal{R}}). \tag{7}$$

According to relationship (7), since ρ_i admit an u.s.r.c, we have

$$\rho_i^+(t,\bar{\mathcal{R}}(t);d_0(t)) < 0, \quad a.e.\ t \in [0,\mathcal{T}], \quad i \in \mathcal{I}.$$

The equation in (6) in turn gives the following strict inequality

$$\rho^+(t,\bar{\mathcal{R}}(t);d_0(t)) < 0, \quad a.e.\ t \in [0,\mathcal{T}]. \tag{8}$$

Indeed, if $\lambda_0 = 0$, then we have

$$0 \le \int_0^{\mathcal{T}} \lambda(t)\rho^+(t,\bar{\mathcal{R}}(t);d(t))dt, \quad \forall d \in L_r^\infty[0,\mathcal{T}].$$

Accordingly, by the Generalized Gordan Theorem, there is no $d \in L_r^\infty[0,\mathcal{T}]$ such that

$$\rho^+(t,\bar{\mathcal{R}}(t);d(t)) < 0, \quad a.e.\ t \in [0,\mathcal{T}].$$

This is a contradiction with (8). So, $\lambda_0 \neq 0$. We put $\mu(t) = \frac{\lambda(t)}{\lambda_0}$, and hence we obtain

$$\begin{aligned}
0 &\le \int_0^{\mathcal{T}} \theta^+(t,\bar{\mathcal{R}}(t);d(t))dt + \int_0^{\mathcal{T}} \mu(t)\rho^+(t,\bar{\mathcal{R}}(t);d(t))dt \\
&\le \int_0^{\mathcal{T}} \Big[\sup_{\xi(t)\in \mathrm{co}\partial^*\theta(t,\bar{\mathcal{R}}(t))} \langle \xi(t), d(t)\rangle + \mu(t) \sup_{\eta(t)\in \mathrm{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))} \langle \eta(t), d(t)\rangle \Big] dt \\
&= \int_0^{\mathcal{T}} \Big[H_{\mathrm{co}\partial^*\theta(t,\bar{\mathcal{R}}(t))}(d(t)) + \mu(t) H_{\mathrm{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))}(d(t)) \Big] dt \\
&= \int_0^{\mathcal{T}} \Big[H_{\{\mathrm{co}\partial^*\theta(t,\bar{\mathcal{R}}(t)) + \mu(t)\mathrm{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))\}}(d(t)) \Big] dt,
\end{aligned}$$

for every $d \in L_r^\infty[0,\mathcal{T}]$. Insomuch the above inequality holds for every $d \in L_r^\infty[0,\mathcal{T}]$, it holds, in special, for constant functions $d(t) = u \in L_r^\infty[0,\mathcal{T}]$.

Sine $\partial^*\theta(t,\bar{\mathcal{R}}(t))$ and $\partial^*\rho(t,\bar{\mathcal{R}}(t))$ are bounded upper semi-regular convexificators, it can be easily verified that the multi-function

$$t \longrightarrow \text{co}\partial^*\theta(t,\bar{\mathcal{R}}(t)) + \mu(t)\text{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))$$

is integrably bounded and compact-valued. By Theorem 1, we have

$$\begin{aligned} H_{\{0\}}(d) = 0 \leq \int_0^{\mathcal{T}} \Big[H_{\{\text{co}\partial^*\theta(t,\bar{\mathcal{R}}(t))+\mu(t)\text{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))\}}(d)\Big]dt \\ = H_{\{\int_0^{\mathcal{T}}[\text{co}\partial^*\theta(t,\bar{\mathcal{R}}(t))+\mu(t)\text{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))]dt\}}(d). \end{aligned}$$

Therefore,

$$0 \in \int_0^{\mathcal{T}} [\text{co}\partial^*\theta(t,\bar{\mathcal{R}}(t)) + \mu(t)\text{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))]dt. \tag{9}$$

It can be inferred from (9) and the definition of integration of multi-functions that there exists a measurable function $\eta(t) \in \text{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))$ a.e. $t \in [0,\mathcal{T}]$ such that

$$0 \in \int_0^{\mathcal{T}} [\text{co}\partial^*\theta(t,\bar{\mathcal{R}}(t)) + \mu(t)\eta(t)]dt. \tag{10}$$

On the other hand,by the Lemma 1, $co\{\bigcup_{i\in I(t,\bar{\mathcal{R}})}\partial^*\rho_i(t,\bar{\mathcal{R}}(t))\}$ is an u.s.r.c for $\rho(t,\cdot)$, we have

$$\eta(t) \in co\{\bigcup_{i\in I(t,\bar{\mathcal{R}})}\partial^*\rho_i(t,\bar{\mathcal{R}}(t))\}.$$

Therefore, there exists $\kappa \in L_r^{\infty}[0,\mathcal{T}]$, $\kappa(t) \geq 0$ a.e. $t \in [0,\mathcal{T}]$ such that

$$\eta(t) \in \sum_{i=1}^{r} \kappa_i(t)\partial^*\rho_i(t,\bar{\mathcal{R}}(t)), \quad \sum_{i=1}^{r}\kappa_i(t) = 1.$$

By defining $\overline{\lambda}_i(t) := \mu(t)\kappa_i(t)$, $i \in \mathcal{I}$, it follows from (10) the proof is complete. □

4 Conclusion

Since the Clarke and Michel-Penot subdifferentials of a locally Lipschitz function are bounded u.s.r.c, the results of Theorem 2 and Lemma 1 in this work are valid with the convexificators replaced by these subdifferentials. Also, the main result of this paper is an actual theorem that might be used to generalize the optimality conditions and duality for non-smooth continuous-time optimization problems and non-smooth multi-objective continuous-time problems in future research.

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Applying Duality Results to Solve the Linear Programming Problems with Grey Parameters

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Abstract. Linear programming problems have exact parameters. In most real-world, we are dealing with situations in which accurate data and complete information are not available. Uncertainty approaches such as fuzzy and random can be used to deal with uncertainties in real-life. Fuzzy and stochastic theories cannot be used if the number of experts and the level of experience is so low that it is impossible to extract membership functions or the number of samples is small. To solve these problems, the grey system theory is proposed. In this paper, a linear programming problem in a grey environment with resources in interval grey numbers is considered. Most of the proposed methods for solving grey linear programming problems become common linear programming problems. However, we seek to solve the problem directly without turning it into a standard linear programming problem for the purpose of maintaining uncertainty in the original problem data in the final solution. For this purpose, we present a method based on the duality theory for solving the grey linear programming problems. This method is more straightforward and less complicated than previous methods. We emphasize that the concept presented is beneficial for real and practical conditions in management and planning problems. Therefore, we shall illustrate our method with some examples in different situations.

Keywords. Grey number, Grey linear programming, Duality theory, Uncertainty.

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1 Introduction

Managers need to use optimization methods to make the right decisions and solve their problems. Operational research is one of the optimization methods that help make the right decisions and solve management problems. One of the standard methods to optimization a goal due to various constraints is Linear Programming (LP) problem [1]. To represent an optimization problem as a linear programming problem, assumptions such as proportionality, additivity, divisibility, and deterministicity of all model parameters are required in the problem formulation [1]. In real cases, these assumptions rarely apply. The LP model is used to select some future activities. As a result, it will inevitably include some degree of uncertainty.

Lack of information, inaccuracy in information, and inaccuracy of forecasting are the characteristics of inaccurate systems. During the decision-making process, the presence of uncertainty in the data and the problem's situation usually confronts the decision-maker with conditions of doubt and uncertainty and makes it difficult to decide and choose the best option [4]. Due to differences in the type and characteristics of uncertainty systems, different theories, methods, and techniques have been used, including statistics and probability, fuzzy set theory, and grey systems theory [16]. Fuzzy set theory is based on the definition of fuzzy numbers, and fuzzy numbers, in turn, depend on the definition of membership functions, which is based on the number of experts and their level of experience. Statistics and probability are also the uncertainty that arises from a completely random process and requires high distribution and sampling functions. In other words, if the decision-maker fails to reduce the uncertainty by obtaining more data, we will face uncertainty from a completely random process.

If the number of experts and level of experience is low and membership functions cannot be extracted, or the number of samples is small i.e., we cannot use fuzzy uncertainty theory or random uncertainty theory; we use grey system theory. This theory, proposed by Deng, provides a very effective way to deal with uncertainty, providing desirable outcomes using low and volatile information [10]. Grey systems theory studies topics that have a definite range and scope and an uncertain nature. With the development of grey systems theory, today, this theory has become a new branch of science whose theoretical structure includes systems analysis, modeling, forecasting, decision making, control, and optimization techniques. Due to this theory's advantages over other methods of dealing with the system of uncertainty, its application is expanding in recent years [2, 5, 34, 35, 38]. LP problems with interval grey numbers have been studied by several authors [3, 9, 14, 15, 19, 23, 27, 36]. For example, Nasseri et al. [24] have proposed a Simplex algorithm-based method for solving Grey Linear Programming (GLP) problem (grey parameter in the objective function) by using grey arithmetic concepts and grey number ranking. Their proposed method has the advantage over previous methods in that it is no longer necessary to whiten GLP problem parameters. Since the problem is solved directly, the input data's uncertainty will be better reflected in the final solution.

The duality theory for inexact LP problems was studied by Soyster [30] and Thuente [32]. Rohn [29] discussed the duality in an interval LP problem with a real-valued objective function. Duality theory was developed by Rodder and Zimmermann [28] for

solving fuzzy parameter LP problems using the aspiration level approach. Several researchers solved fuzzy programming problems using fuzzy duality theory [12, 18, 31]. Ramik [26] introduced some new concepts, and results, possibilities, and necessary relations of duality in fuzzy linear programming. Nasseri and Darvishi [23] gave duality theory for solving the GLP problems. Darvishi [7] has studied some of the duality results in the GLP problem. One of the weaknesses of LP with grey parameters problems is solving the problem without converting the parameters to display the uncertainty of the input data in the output solution. To solve this problem, in this research, we tried to use a dual LP problem with grey data, to present a method without whitening grey parameters to solve the LP problem with grey data.

This research is formatted as follows. Section 2, presents some necessary notations and definitions of grey systems. The definition of the GLP problem is given in Section 3. In Section 4, a dual LP problem with grey parameters and a new algorithm based on dual results to solve it are introduced. In Section 5, by presenting three different examples, the efficiency and numerical analysis of the proposed method is presented. Finally, Section 6 consists of conclusions.

2 Preliminaries

In this section, we describe the definitions and concepts needed to study and analyze the mathematical of grey systems and grey number calculations to solve the GLP problem [10, 13, 17]. Grey systems theory is one of the most critical methods for studying and analyzing systems with incorrect parameters and incomplete information [16]. One of the main concepts of grey systems theory that plays a significant role in studying uncertainty is grey numbers. There are different kinds of grey numbers [16] which we use interval grey numbers in this article.

Definition 1. An interval grey number is the one whose exact value is unknown but whose range is known [6].

$$\otimes x \in [\underline{x}, \overline{x}] = \{t | \underline{x} \leq t \leq \overline{x}\}, \ \underline{x} \leq \overline{x}, \tag{1}$$

where, t is grey number information, $\underline{x}$ lower limit, and $\overline{x}$ upper limit.

Remark 1. [6] We show the set of grey numbers with $R(\otimes)$ the symbol.

Definition 2. [21] Let $\otimes x_1 \in [\underline{x}_1, \overline{x}_1]$ and $\otimes x_2 \in [\underline{x}_2, \overline{x}_2]$ be two grey numbers. The following operations can be defined:

$$\otimes x_1 + \otimes x_2 = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2],$$
$$\otimes x_1 - \otimes x_2 = \otimes x_1 + (-\otimes x_2) = [\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2],$$
$$\otimes x_1 \times \otimes x_2 = \left[\min\{\underline{x}_1\underline{x}_2, \overline{x}_1\overline{x}_2, \overline{x}_1\underline{x}_2, \underline{x}_1\overline{x}_2\}, \max\{\underline{x}_1\underline{x}_2, \overline{x}_1\overline{x}_2, \overline{x}_1\underline{x}_2, \underline{x}_1\overline{x}_2\}\right],$$
$$k \cdot \otimes x \in \begin{cases} [k\underline{x}, k\overline{x}], & k > 0, \\ [k\overline{x}, k\underline{x}], & k < 0. \end{cases}$$

Remark 2. [13] Let $\otimes x \in [\underline{x}, \overline{x}]$ be a grey number. Then we have $\otimes x \div \otimes x = 1$, and $\otimes x - \otimes x = 0$.

Remark 3. [13] For any real number x, we have $\otimes x \in [x, x]$.

Definition 3. [27] For any grey number $\otimes x \in [\underline{x}, \overline{x}]$, the kernel, $\otimes\widehat{x}$ of the grey number is defined as $\otimes\widehat{x} = \dfrac{\underline{x} + \overline{x}}{2}$.

Definition 4. [37] The length of the grey number $\otimes x \in [\underline{x}, \overline{x}]$ is defined as $\mu(\otimes x) = \overline{x} - \underline{x}$.

Ranking of grey numbers is very important in grey decision making and optimization problems. Several methods and more details can be seen in Dervishi et al.[8].

Definition 5. [37] Suppose that the background, which makes a grey number $\otimes x$ come into being, is Ω and $\mu(\Omega)$ is the value of Ω. Then $g^{\circ}(\otimes x) = \dfrac{\mu(\otimes x)}{\mu(\Omega)}$ is called the degree of greyness of $\otimes x$ (denoted as g° for short).

Definition 6. [11] Suppose $\otimes x_1$ and $\otimes x_2$ are two grey numbers and $\otimes\widehat{x_1}$, $\otimes\widehat{x_2}$ are the center of $\otimes x_1$ and $\otimes x_2$ respectively, $g^{\circ}(\otimes x_1)$ and $g^{\circ}(\otimes x_2)$ are the degree of greyness of $\otimes x_1$ and $\otimes x_2$, respectively. So, if $\otimes\widehat{x_1} < \otimes\widehat{x_2} \Rightarrow \otimes x_1 <_G \otimes x_2$

$$\text{if} \qquad \otimes\widehat{x_1} = \otimes\widehat{x_2} \Rightarrow \begin{cases} \text{if} & g^{\circ}(\otimes x_1) = g^{\circ}(\otimes x_2) \Rightarrow \otimes x_1 =_G \otimes x_2, \\ \text{if} & g^{\circ}(\otimes x_1) < g^{\circ}(\otimes x_2) \Rightarrow \otimes x_1 >_G \otimes x_2, \\ \text{if} & g^{\circ}(\otimes x_1) > g^{\circ}(\otimes x_2) \Rightarrow \otimes x_1 <_G \otimes x_2. \end{cases} \tag{2}$$

For further study on the grey theory systems see [13].

3 Grey Linear Programming

The GLP problem is one of the appropriate approaches to deal with uncertainty in real-life problems. Here, we present the general model of LP problems, including grey numbers

$$\begin{aligned} \text{Maximize} \otimes Z =_G & \sum_{j=1}^{n} \otimes c_j \otimes x_j \\ \textbf{s.t} \qquad & \sum_{j=1}^{n} \otimes \otimes a_{ij} x_j \leq_G b_i, \\ & \otimes x_j \geq_G \otimes 0, \end{aligned} \tag{3}$$

where $\otimes c_j, \otimes a_{ij}, \otimes x_j, \otimes b_i \in R(\otimes), i = 1, 2, ..., m,\ j = 1, 2, ..., n.$

Definition 7. Linear programming problem with grey right-hand sides' parameters is defined as follows:

$$\begin{aligned} \text{Minimize} \otimes Z =_G & \sum_{j=1}^{n} \otimes c_j \otimes x_j \\ \textbf{s.t} \quad & \sum_{j=1}^{n} \otimes \otimes a_{ij} x_j \geq_G b_i, \\ & \otimes x_j \geq_G \otimes 0, \end{aligned} \tag{4}$$

where $\otimes c_j, \otimes a_{ij} \in R, \otimes x_j, \otimes b_i \in R(\otimes), i = 1, 2, ..., m,\ j = 1, 2, ..., n.$

Because in the process, we need the notion of a feasible grey solution and grey optimal solution, we consider the following definitions.

Definition 8. The set $\{\otimes x_j,\ j = 1, 2, ..., n\}$ is called to be a feasible solution (4), if they satisfy into the model constraints.

Definition 9. A feasible solution $\{\otimes x_j,\ j = 1, 2, ..., n\}$ of the problem (4), is said to be an optimal feasible solution, if $\sum_{j=1}^{n} \otimes c_j \otimes x_0 \leq \sum_{j=1}^{n} \otimes c_j \otimes x_j,\ \otimes x_j \in D,\ j = 1, 2, ..., n.$

Definition 10. (Grey basic feasible solution): Consider the system $\sum_{j=1}^{n} a_{ij} \otimes x_j =_G \otimes b_i\, i = 1, 2, ..., m$ and $\otimes x_j \geq_G 0$, where $[a_{ij}]_{m \times n}$ is a $m \times n$ matrix and $[\otimes b_i] i = 1, 2, ..., m$ is an m vector. Suppose that $rank[a_{ij}, \otimes b_i]_{m \times n+1} = rank[a_{ij}]_{m \times n} = m$ partition, $A = [a_{ij}]_{m \times n}$, $j = 1, 2, ..., n$, $i = 1, 2, ..., m$, after possibly rearranging the columns of A, as $[B, N]$, where B is $m \times m$ non-singular matrix. It is apparent that

$$\otimes x_B =_G (\otimes x_{B_1}, \otimes x_{B_2}, ..., \otimes x_{B_m})^T =_G B^{-1} \otimes b, \quad \otimes x_N =_G \otimes 0,$$

is a solution of $\sum_{j=1}^{n} a_{ij} \otimes x_j =_G \otimes b_i,\ i = 1, 2, ..., m$ the vector $\otimes x =_G (\otimes x_B^T, \otimes x_N^T)^T$ where $\otimes x_N =_G \otimes 0$ is called a basic grey solution of the system.

If $\otimes x_B \geq_G \otimes 0$, then $\otimes x$ is called a grey basic feasible solution of the system, and the corresponding grey objective value is $\otimes z =_G c_B \otimes x_B$, where $c_B = \left(c_{B_1}, c_{B_2}, ..., c_{B_m}\right)$.

For all $j = 1, 2, ..., n$, define $y_j = B^{-1} a_j$, $z_j = c_B y_j = c_B B^{-1} a_j$ and for any primary index $j = B_i,\ j = 1, 2, ..., m$, we have $z_j - c_j = c_B B^{-1} a_j - c_j = 0$.

Theorem 1. Let problem (4) be non-degenerate. A basic feasible solution, $\otimes x_B =_G (\otimes x_{B_1}, \otimes x_{B_2}, ..., \otimes x_{B_m})^T =_G B^{-1} \otimes b$, $\otimes x_N =_G \otimes 0$ is optimal to (4) if and only if, $z_j - c_j = c_B B^{-1} a_j - c_j \leq 0$, for any non-basic variable.

Proof. Let that $\otimes x =_G (\otimes x_B^T, \otimes x_N^T)^T$, $\otimes x_B =_G (\otimes x_{B_1}, \otimes x_{B_2}, ..., \otimes x_{B_m})^T =_G B^{-1} \otimes b$, $\otimes x_N =_G \otimes 0$ is a solution of $\sum_{j=1}^{n} a_{ij} \otimes x_j =_G \otimes b_i,\ i = 1, 2, ..., m$. So that, the optimal grey value of the objective function is $\otimes Z =_G c_B \otimes x_B =_G c_B B^{-1} \otimes b$. On the other hand, for any grey basic feasible solution $\otimes x$ to (4), we have $\otimes b =_G A \otimes x =_G B \otimes x_B + N \otimes x_N$. Hence, $B \otimes x_B =_G \otimes b - N \otimes x_N \Rightarrow \otimes x_B =_G B^{-1} \otimes b - B^{-1} N \otimes x_N$.

Thus, for any basic grey feasible solution to (4), we have

$$\otimes Z =_G c \otimes x =_G c_B \otimes x_B + c_N \otimes x_N,$$

$$
\begin{aligned}
&=_G c_B B^{-1} \otimes b - c_B B^{-1} N \otimes x_N + c_N \otimes x_N, \\
&=_G c_B B^{-1} \otimes b - \left(c_B B^{-1} N - c_N\right) \otimes x_N, \\
&=_G c_B B^{-1} \otimes b - \sum_{j \neq B_i} \left(c_B B^{-1} a_j - c_j\right) \otimes x_j, \\
&=_G c_B B^{-1} \otimes b - \sum_{j \neq B_i} \left(z_j - c_j\right) \otimes x_j.
\end{aligned}
$$

Now, consider the following three cases:
1) If for all $j = 1, 2, ..., n$, we have:

$$
\left(z_j - c_j\right) \geq 0 \Rightarrow \left(z_j - c_j\right) \otimes x \geq_G \otimes 0 \Rightarrow \sum_{j \neq B_i} \left(z_j - c_j\right) \otimes x_j \geq_G \otimes 0, \text{then} \otimes Z \leq_G \otimes Z^*.
$$

This is a contradiction to $\otimes Z^*$ being optimal.
2) If for all $j = B_i, 1 \leq i \leq m$, we have $\left(z_j - c_j\right) = 0$, then $\otimes Z^* =_G \otimes Z$.
3) If for all $j = 1, 2, ..., n$, we have:

$$
\left(z_j - c_j\right) \leq 0 \Rightarrow \left(z_j - c_j\right) \otimes x \leq_G \otimes 0 \Rightarrow \sum_{j \neq B_i} \left(z_j - c_j\right) \otimes x_j \leq_G \otimes 0, \text{then} \otimes Z \geq_G \otimes Z^*.
$$

Hence, we have $z_j - c_j = c_B B^{-1} a_j - c_j \leq 0$, $j = 1, 2, ..., n$. □

4 Duality in Linear Programming Problem with Grey Parameters

Duality is an essential concept in linear algebra and mathematical programming that derives from examining a problem from two different perspectives. Specifically, a device is interpreted from linear relationships defined in terms of a matrix in terms of column space or its row space. These two different perspectives lead us to some real results, such as the equation of the row and column rank of a matrix, the equation of optimal values of the initial problem, and the dual problem of LP [20]. The duality concept is one of the most essential and exciting concepts in linear programming. The basic idea in this theory is that every LP problem has a corresponding dual problem. So, whenever an LP problem is given, by solving it in the simplex method, we have two problems solution synchronous. For each LP problem with grey parameters, there is a corresponding problem named dual, which satisfies the duality results and the crisp and fuzzy environments [23]. Therefore, we try to find a solution to the LP problems with grey parameters using the duality concept. If we determine the dual of the problem (4), then we can present the mathematical model of the LP problem by grey right-hand sides as follows:

$$
\begin{aligned}
&\text{Minimize} \quad \otimes Z =_G \sum_{j=1}^{m} \otimes y_i \otimes b_i \\
&\mathbf{s.t} \qquad \sum_{j=1}^{m} \otimes \otimes y_i a_{ij} \geq_G c_j, \; j = 1, 2, ..., n,
\end{aligned} \tag{5}
$$

$$\otimes y_i \geq_G \otimes 0 \;\; i = 1, 2, ..., m,$$

where $c_j, a_{ij} \in R$, $y_i, b_i \in R(\otimes), i = 1, 2, ..., m, \;\; j = 1, 2, ..., n.$

Theorem 2. [23] (The weak duality property) If $\otimes x^0 = \left(\otimes x_1^0, \otimes x_2^0, ..., \otimes x_n^0\right) \geq_G \otimes 0$ is any feasible solution to the primal GLP problem (4) and $\otimes y^0 = (\otimes y_1^0, \otimes y_2^0, ..., \otimes y_m^0) \geq_G \otimes 0$ is any feasible solution to the dual of the problem (4) i.e. (5), then $\sum_{i=1}^{m} \otimes y_i^0 \otimes b_i, \leq_G \sum_{j=1}^{n} c_j \otimes x_j^0$.

Corollary 1. [23] If $\otimes x^*$ and $\otimes y^*$ are, respectively, feasible solutions to primal (4) and dual (5), and $c \otimes x^* =_G \otimes y^{*T} \otimes b$, then $\otimes x^*$ and $\otimes y^*$are optimal solutions to their respective problems.

4.1 The proposed algorithm (Duality Method)

Now we propose an a Algorithm based on the duality method for finding the optimal solution. Consider the following problem:

$$\begin{aligned} \text{Minimize} \;\; & \otimes Z =_G C \otimes X \\ \textbf{s.t} \quad & A \otimes X \geq_G \otimes b, \\ & \otimes X \geq_G \otimes 0. \end{aligned} \tag{6}$$

1) Write a dual problem of grey linear programming.

$$\begin{aligned} \text{Maximize} \;\; & \otimes u =_G \otimes y^T \otimes b \\ \textbf{s.t} \quad & \otimes y^T A \leq_G C, \\ & \otimes y^T \geq_G \otimes 0. \end{aligned} \tag{7}$$

2) Suppose that a basic feasible solution and a corresponding simplex table are available.

3) The basic feasible solution is given by $\otimes y_B^T =_G B^{-1} c =_G \otimes f_0$ and $\otimes y_N^T =_G \otimes 0$. In this case, the value of the grey target function will be as: $\otimes u =_G \otimes y^T B^{-1} C =_G \otimes f_{00}$.

4) Compute $\otimes f_{0j} =_G \otimes u_j - \otimes b_j$, for all $j = 1, 2, ..., m$; $j \neq B_i$, $i = 1, 2, ..., n$.

5) Let $\otimes f_{0k} =_G \min\left\{\otimes f_{ij}\right\}, \;\; j = 1, 2, ..., m.$

6) If $\otimes f_{0k} \geq_G \otimes 0$, then stop, the current solution is optimal.

7) If $\otimes f_{0k} <_G \otimes 0$ and, $f_{ik} \leq \otimes 0$, $i = 1, 2, ..., n$, then stop, the problem is unbounded.

8) If $\otimes f_{0k} <_G \otimes 0$ and, $f_{ik} > \otimes 0$, $i = 1, 2, ..., n$, then, determine an index r corresponding to a variable y_{B_r} leaving the basis as follows:

$$\frac{f_{r0}}{f_{rk}} = \min\left\{\frac{f_{i0}}{f_{ik}} | f_{ik} \geq 0\right\}, i = 1, 2, ..., n.$$

9) Pivot on f_{rk}an element and update the simplex tableau to go to the second step.

In the following, the efficiency of the proposed method is shown by providing practical examples.

5 Numerical Examples

In this section, by presenting three different examples of problem (4), including different types of constraints and decision variables that can occur in real life, the proposed method's efficiency to solve them is evaluated. Here, we give GLP problems and solve these by the proposed method described in the last section.

Example 1. A livestock company is willing to provide the feed required by its livestock at a minimum cost. The number of nutrients in each kilogram of these substances (in terms of the number of units of nutrients in the substance), the number of nutrients needed per day, and each ingredient's cost are listed below.

Table 1: The nutrients, cost, and required daily amount of a Livestock company.

Amount of daily necessities	Alfalfa	Corn	Nutrients
$\otimes[4,6]$	3	2.5	Vitamin
$\otimes[3,4]$	4	1	Protein
	4	2	Cost

Let
$\otimes x_1$: Number of packages needed for Corn;
$\otimes x_2$: Number of packages needed for Alfalfa.
Consider the following GLP problem.

$$\begin{aligned} \text{Minimize } & \otimes Z =_G 2 \otimes x_1 + 4 \otimes x_2 \\ \textbf{s.t} \quad & 2.5 \otimes x_1 + 3 \otimes x_2 \geq_G \otimes[4,6], \\ & \otimes x_1 + 4 \otimes x_2 \geq_G \otimes[3,4], \\ & \otimes x_1, \otimes x_2 \geq_G \otimes 0, \end{aligned} \tag{8}$$

and its dual problem:

$$\begin{aligned} \text{Maximize } & \otimes u =_G \otimes[4,6] \otimes y_1 + \otimes[3,4] \otimes y_2 \\ \textbf{s.t} \quad & 2.5 \otimes y_1 + \otimes y_2 \leq_G 2, \\ & 3 \otimes y_1 + 4 \otimes y_2 \leq_G 4, \\ & \otimes y_1, \otimes y_2 \geq_G \otimes 0. \end{aligned} \tag{9}$$

In this example, we describe a primal model to the constraints of the "greater than or equal to" type, the decision variables are "≥ 0", and so, duality model has the

constraints of the "less than or equal to" type. Also, the decision variables are "≥ 0". Now, we solve the problem by using the proposed Algorithm 4.1.

Table 2: The simplex tableau of the problem (9).

Basic variables	$\otimes u$	$\otimes y_1$	$\otimes y_2$	$\otimes s_1$	$\otimes s_2$	R.H.S
$\otimes u_0$	[1,1]	-[4,6]	-[3,4]	[0,0]	[0,0]	[0,0]
$\otimes s_1$	[0,0]	2.5	1	[1,1]	[0,0]	2
$\otimes s_2$	[0,0]	3	4	[0,0]	[1,1]	4
$\otimes u_0$	[1,1]	[0,0]	-[0.6,2.4]	[1.6,2.4]	[0,0]	[3.2,4.8]
$\otimes y_1$	[0,0]	[1,1]	0.4	0.4	[0,0]	0.8
$\otimes s_2$	[0,0]	[0,0]	2.8	-1.2	[1,1]	1.6
$\otimes u_0$	[1,1]	[0,0]	[0,0]	[0.57,2.14]	[0.21,0.85]	[3.54,6.17]
$\otimes y_1$	[0,0]	[1,1]	[0,0]	0.57	0.14	0.571
$\otimes y_2$	[0,0]	[0,0]	[1,1]	0.43	0.36	0.571

Table 2 is the optimal table of the duality problem, so by using the coefficients of the slack variables of the duality problem in the first line of the optimal table, we extract the optimal answer to the primal problem. Grey optimal solution to the primal problem will be as follows.

$$\otimes x_1 = \otimes[0.57, 2.14], \quad \otimes x_2 = \otimes[0.21, 0.85], \quad \otimes z = \otimes[3.54, 6.17].$$

Example 2. Consider the following GLP problem:

$$\begin{aligned} \text{Minimize } & \otimes Z =_G \otimes x_1 + 2 \otimes x_2 + \otimes x_3 \\ \textbf{s.t} \quad & 2 \otimes x_1 + 3 \otimes x_2 + \otimes x_3 \leq_G \otimes[4, 8], \\ & 2 \otimes x_1 + 3 \otimes x_2 + \otimes x_3 \geq_G \otimes[0.5, 1.5], \\ & \otimes x_1, \otimes x_2, \otimes x_3 \geq_G \otimes 0. \end{aligned} \tag{10}$$

and its dual problem:

$$\begin{aligned} \text{Maximize } & \otimes u =_G \otimes[4, 8] \otimes y_1 + \otimes[0.5, 1.5] \otimes y_2 \\ \textbf{s.t} \quad & -2 \otimes y_1 + 2 \otimes y_2 \leq_G 1, \\ & 3 \otimes y_1 + 3 \otimes y_2 \leq_G 2, \\ & -\otimes y_1 + \otimes y_2 \leq_G 1, \\ & \otimes y_1, \otimes y_2 \geq_G \otimes 0. \end{aligned} \tag{11}$$

In this example, we describe a primal model contain some constraints of the "less than or equal to" type, some of the "greater than or equal to" type, the decision variables are "≥ 0". So, the duality model has the constraints of the "less than or equal to" type, and the decision variables are "≥ 0". Now, we solve the problem according to the Algorithm 4.1.

Table 3 gives the optimal table of the duality problem, so by using the coefficients of the slack variables of the duality problem in the first line of the optimal table, we

Table 3: The simplex tableau of the problem (11).

Basic variables	$\otimes u$	$\otimes y_1$	$\otimes y_2$	$\otimes s_1$	$\otimes s_2$	$\otimes s_3$	R.H.S
$\otimes u_0$	[1,1]	[4,8]	-[0.5,1.5]	[0,0]	[0,0]	[0,0]	[0,0]
$\otimes s_1$	[0,0]	-2	2	1	0	0	1
$\otimes s_2$	[0,0]	3	3	0	1	0	2
$\otimes s_3$	[0,0]	-1	1	0	0	1	1
$\otimes u_0$	[1,1]	[2.5,7.5]	[0,0]	[0.25,0.75]	[0,0]	[0,0]	[0.25,0.75]
$\otimes y_2$	[0,0]	-1	1	0.5	0	0	0.5
$\otimes s_2$	[0,0]	6	0	-1.5	1	0	0.5
$\otimes s_3$	[0,0]	-1	1	-0.5	0	1	0.5

extract the optimal solution of the primal problem. The solution of the primal problem will be as follows:

$$\otimes x_1 = \otimes[0.25, 0.75],\ \otimes x_2 = \otimes[0, 0],\ \ \otimes x_3 = \otimes[0, 0],\ \ \otimes Z = \otimes[0.25, 0.75].$$

Example 3. Consider the following GLP problem.

$$\begin{aligned} \text{Minimize } \otimes Z =_G & 10 \otimes x_1 + 50 \otimes x_2 + 20 \otimes x_3 \\ \textbf{s.t} \quad & \otimes x_1 + 2 \otimes x_2 + \otimes x_3 \geq_G \otimes[400, 600], \\ & \otimes x_1 + 8 \otimes x_2 \geq_G \otimes[100, 300], \\ & \otimes x_2, \otimes x_3 \geq_G \otimes 0,\ \otimes x_1 \text{ is unrestricted}, \end{aligned} \tag{12}$$

and its dual problem:

$$\begin{aligned} \text{Maximize } \otimes u =_G & \otimes[400, 600] \otimes y_1 + \otimes[100, 300] \otimes y_2 \\ \textbf{s.t} \quad & \otimes y_1 + \otimes y_2 =_G 10, \\ & 2 \otimes y_1 + 8 \otimes y_2 \leq_G 50, \\ & \otimes y_1 \leq_G 20, \\ & \otimes y_1, \otimes y_2 \geq_G \otimes 0. \end{aligned} \tag{13}$$

In this example, we describe a primal model to constraints of the "greater than or equal to" type, the decision variables are "$\geq_G 0$", or "unrestricted". So, the duality model has the constraints of the "less than or equal to" type, and some of the "equal to" type, and the decision variables are "$\geq_G 0$". Now, we solve the problem according to Algorithm 4.1.

Table 4 presents the optimal table of the duality problem, so by using the coefficients of the slack variables of the duality problem in the first line of the optimal table, we extract the optimal answer to; the primal problem. The solution to the primal problem will be as follows.

$$x_1 = \otimes[400, 600],\ \ \otimes x_2 = \otimes[0, 0],\ \ \otimes x_3 = \otimes[0, 0],\ \otimes Z = \otimes[4000, 6000].$$

Different methods have been proposed to solve GLP problems, most of which have been using whitening problem parameters. GLP problems were studied by several authors

Table 4: The simplex tableau of the problem (13).

Basic variables	$\otimes u$	$\otimes y_1$	$\otimes y_2$	$\otimes R_1$	$\otimes s_2$	$\otimes s_3$	R.H.S
$\otimes u_0$	[1,1]	-[400,600]	-[100,300]	[M,M]	[0,0]	[0,0]	[0,0]
$\otimes R_1$	[0,0]	1	1	[1,1]	[0,0]	[0,0]	10
$\otimes s_2$	[0,0]	2	8	[0,0]	[1,1]	[0,0]	50
$\otimes s_3$	[0,0]	1	0	[0,0]	[0,0]	[1,1]	20
$\otimes u_0$	[1,1]	-[M,M] -[400,600]	-[M,M] -[100,300]	[0,0]	[0,0]	[0,0]	-30[M,M]
$\otimes R_1$	[0,0]	1	1	[1,1]	[0,0]	[0,0]	10
$\otimes s_2$	[0,0]	2	8	[0,0]	[1,1]	[0,0]	50
$\otimes s_3$	[0,0]	1	0	[0,0]	[0,0]	[1,1]	20
$\otimes u_0$	[1,1]	[0,0]	[100,500]	[M,M] +[400,600]	[0,0]	[0,0]	+10[M,M] +[400,600]
$\otimes y_1$	[0,0]	[1,1]	1	1	[0,0]	[0,0]	10
$\otimes s_2$	[0,0]	[0,0]	6	-2	[1,1]	[0,0]	30
$\otimes s_3$	[0,0]	[0,0]	-1	-1	[0,0]	[1,1]	10

[9, 14, 15, 19, 27, 33, 36]. Nasseri et al. [24] presented the first method of solving GLP problems without whitening, which had grey objective function coefficients for GLP problems. In this paper, we have presented for the first time a method for solving GLP problems with the right-hand side grey parameters without whitening grey numbers. In this method, we use the dual GLP problem that has not been done before, and we do so for constraints in different states. The presented Algorithm for this method is straightforward and will show the uncertainty of the input data in the results.

6 Conclusion

Grey systems theory is a fundamental methodology for dealing with inexact conditions. The problem of LP with grey resources is beneficial for real and practical problems. In this research, we fixed a new concept of duality for LP minimization problems with grey parameters. We argue the grey basic feasible solution notions for LP minimization problems with grey parameters. With the use of arithmetic operations between interval grey numbers, we have proved the optimal basic feasible solution and the weak duality property for LP minimization problems with grey data. These results would be useful for establishing a new Algorithm. Finally, by giving different examples, we demonstrate the proposed method's efficiency for solving GLP in different modes. In the most proposed methods for find LP problems solution with grey parameters, the GLP problems transformed into one or a series of the classical LP problems and then obtained an optimal solution, but with the use of the above results, we proposed a new Algorithm (duality method) for the find LP problems solution with grey data. The proposed method is less sophisticated than other methods. One of the essential advantages of this method is that it does not use whitening to solve GLP problems, resulting presented uncertainty in the input data in the output data.

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A Fuzzy Distance Measure for Fuzzy Numbers

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Abstract. A fuzzy distance measure is introduced in this paper to evaluate the fuzzy distance between two fuzzy numbers. For this purpose, α-values of fuzzy numbers are used to develop an integral-based fuzzy distance measure. The properties of the proposed fuzzy distance measure are verified. The proposed fuzzy distance measure is also compared with other fuzzy distance measures.

Keywords. Fuzzy number, α-values, Fuzzy distance measure, Robustness.

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1 Introduction

Fuzzy sets [47] have been employed in many real-life applications. In many applications often there is need a distance measure between two fuzzy numbers to solve the problems. Many distance measures have been proposed for fuzzy sets including exact [1, 6, 8, 14, 20, 21, 26, 27, 29, 31, 32, 33, 36, 40] and fuzzy quantities [3, 4, 10, 11, 18, 24, 22, 34, 37, 41]. The idea behind a fuzzy distance is that if the quantities are reported as fuzzy numbers, their distance is also expected to be a fuzzy number. Other distance measures were conducted based on other types of imprecision including intuitionistic fuzzy sets and interval-valued fuzzy numbers [15, 17, 19, 28, 39, 42, 43, 44, 45, 48, 49, 5].

A distance measure is an essential issue in many real-life applications that involve error measurements between non-exact quantities. Many studies, as mentioned before, have calculated the distance between two fuzzy numbers as exact numbers. However, an exact distance is not usually reasonable in the fuzzy domain since an exact value for distance between two fuzzy numbers may outcome in loss of essential information under imprecision. Thus, it is reasonable to say that the distance between two fuzzy numbers should be expressed as a fuzzy number, too. Furthermore, many existing fuzzy distances do not satisfy some proper properties of a distance measure expected in the fuzzy domain. In this paper, a new fuzzy distance measure was proposed for fuzzy numbers. It was shown that the proposed fuzzy distance measure satisfied all properties for an absolute error distance in the fuzzy domain. The main advantages of the proposed fuzzy distance measure over others were also extensively illustrated.

This paper is organized as follows: Section 2 reviews some concepts of fuzzy numbers. Section 3 introduces fuzzy distance measures between two fuzzy numbers. The main properties of the proposed fuzzy distance measures are also discussed and compared with other fuzzy distances. The main contributions of this study will be discussed in Section 4.

2 Fuzzy Numbers

This section briefly reviews some concepts and terminology related to α-values of fuzzy numbers used throughout this paper.

Let $\mathbb{X}$ be a universal set. A fuzzy set of $\widetilde{A}$ is defined by its membership function $\widetilde{A} : \mathbb{X} \to [0,1]$. The set $\widetilde{A}[\alpha] := \{x \in \mathbb{X} : \widetilde{A}(x) \geq \alpha\}$ is called the α-cut of $\widetilde{A}$ [30]. $\widetilde{A}$ is called a fuzzy number (**FN**) on $\mathbb{X} = \mathbb{R}$ if

1. There exists a unique $x^*_{\widetilde{A}} \in \mathbb{R}$ with $\widetilde{A}(x^*_{\widetilde{A}}) = 1$, and
2. the set $\widetilde{A}[\alpha] = \{x \in \mathbb{R} : \widetilde{A}(x) \geq \alpha\}$ is a non-empty nested closed interval in $\mathbb{R}$, for every $\alpha \in (0,1]$.

Such interval is presented by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$ in which $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. Moreover, a fuzzy number of $\widetilde{A}$ is an LR-fuzzy number (**LRFN**) if there exist real numbers of a, l_a and r_a with $l_a, r_a \geq 0$, and strictly decreasing and

continuous functions of $L, R : [0,1] \to [0,1]$ such that

$$\widetilde{A}(x) = \begin{cases} L(\frac{a-x}{l_a}), & a - l_a \le x \le a, \\ R(\frac{x-a}{r_a}), & a < x \le a + r_a, \\ 0, & x \in \mathbb{R} - [a - l, a + r]. \end{cases} \tag{1}$$

where $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. An LR-fuzzy number of $\widetilde{A}$ can be simply denoted by $(a; l_a, r_a)_{LR}$. The most commonly used LR-fuzzy numbers are triangular fuzzy numbers (**TFN**s) in which the shape functions of L and R are given by $L(x) = R(x) = 1 - x$, for all $x \in [0,1]$. The membership function of a **TFN** of $\widetilde{A} = (a; l, r)_T$ is denoted by

$$\widetilde{A}(x) = \begin{cases} \frac{x-a+l_a}{l_a}, & a - l_a \le x \le a, \\ \frac{a+r_a-x}{r_a}, & a \le x \le a + r_a, \\ 0, & x \in \mathbb{R} - [a - l_a, a + r_a]. \end{cases} \tag{2}$$

Some common operations between two LR-fuzzy numbers of $\widetilde{A} = (a; l_a, r_a)_{LR}$ and $\widetilde{B} = (b; l_b, r_b)_{LR}$ can be defined as follows [30]:

1) (Addition) $\widetilde{A} \oplus \widetilde{B} = (a + b; l_a + l_b, r_a + r_b)_{LR}$.

3) (Scalar multiplication):

$$\lambda \otimes \widetilde{A} = \begin{cases} (\lambda a; \lambda l_a, \lambda r_a)_{LR}, & if \ \lambda > 0, \\ (\lambda a; -\lambda r_a, -\lambda l_a)_{RL}, & if \ \lambda < 0. \end{cases} \tag{3}$$

Here, the notion of α-values of **FN**s is recalled.

Definition 1. [23] The α-values of a **FN** $\widetilde{A}$ is a mapping $\widetilde{A}_\alpha : [0,1] \to \mathbb{R}$ defined by:

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^L_{2\alpha}, & \alpha \in [0, 0.5], \\ \widetilde{A}^U_{2(1-\alpha)}, & \alpha \in [0.5, 1], \end{cases} \tag{4}$$

where $\widetilde{A}^L_\alpha$, $\widetilde{A}^U_\alpha$ show the lower and upper limits of $\widetilde{A}[\alpha]$, respectively.

Example 1. Let $\widetilde{A} = (a; l_a, r_a)_{LR}$ be an LR-**FN**. From Definition 1, one finds that:

$$\widetilde{A}_\alpha = \begin{cases} a - l_a L^{-1}(2\alpha), & 0 \le \alpha \le 0.5, \\ a + r_a R^{-1}(2(1-\alpha)), & 0.5 \le \alpha \le 1. \end{cases}$$

For instance,

1. If $\widetilde{A} = (a; l_a, r_a)_T$ is a **TFN**, then,

$$\widetilde{A}_\alpha = \begin{cases} (a - l_a) + 2l_a\alpha, & 0 \le \alpha \le 0.5, \\ a + r_a - 2r_a(1-\alpha), & 0.5 \le \alpha \le 1. \end{cases}$$

2. Let $\widetilde{A} = (a; l_a, r_a)_{LR}$ with $L(x) = \sqrt{1-x^3}$ and $R(x) = 1 - x^5$ then:

$$\widetilde{A}_\alpha = \begin{cases} a - l_a \sqrt[3]{1-4\alpha^2}, & 0 \le \alpha \le 0.5, \\ a + l_a \sqrt[5]{2\alpha - 1}, & 0.5 \le \alpha \le 1. \end{cases}$$

The relationship between α-values and α-cuts of **FN**s can be investigated by the following lemma [23].

Lemma 1. Let $\{\widetilde{A}_\alpha\}_{\alpha\in[0,1]}$, called α-values, be a strictly decreasing function of α and $\widetilde{A}_{0.5}$ be a constant number. Then, $\{\widetilde{A}_\alpha\}_{\alpha\in[0,1]}$ can construct a **FN** $\widetilde{A}$ whose α-cuts is $\widetilde{A}[\alpha] = [\widetilde{A}_{1-\alpha/2}, \widetilde{A}_{\alpha/2}]$. In addition, if $\widetilde{A}$ is a **FN**, then its α-values are given by:

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^L_{2\alpha}, & \alpha \in [0, 0.5], \\ \widetilde{A}^U_{2(1-\alpha)}, & \alpha \in (0.5, 1]. \end{cases} \tag{5}$$

Proof. Given the α-values of $\{\widetilde{A}_\alpha\}_{\alpha\in[0,1]}$, it is easy to verify that $[\widetilde{A}_{1-\alpha_2/2}, \widetilde{A}_{\alpha_2/2}] \subseteq [\widetilde{A}_{1-\alpha_1/2}, \widetilde{A}_{\alpha_1/2}]$. Therefore, from Representation Theorem [30], the α-cuts of $\widetilde{A}[\alpha] = [\widetilde{A}_{1-\alpha/2}, \widetilde{A}_{\alpha/2}]$ can construct the fuzzy number of $\widetilde{A}$. Therefore, $\widetilde{A}$ is a **FN**. Now, let $\widetilde{A}$ be a **FN** with the following α-values:

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^U_{2\alpha}, & \alpha \in [0, 0.5], \\ \widetilde{A}^L_{2(1-\alpha)}, & \alpha \in (0.5, 1]. \end{cases} \tag{6}$$

Then, it is readily seen that

1) $\widetilde{A}_\alpha$ is a strictly decreasing function with relative to $\alpha \in [0, 1]$ and $\widetilde{A}_{0.5}$ is a constant number,

2) $\widetilde{A}[\alpha] = [\widetilde{A}_{1-\alpha/2}, \widetilde{A}_{\alpha/2}]$,

which completes the proof. □

Definition 2. [25] Let $\widetilde{A}$ and $\widetilde{B}$ be two **FN**s. It is said that $\widetilde{A} \preceq \widetilde{B}$, if $\widetilde{A}_\alpha \le \widetilde{B}_\alpha$ for any $\alpha \in [0, 1]$.

We will use such ordering to define a fuzzy distance measure in the next section.

Remark 1. [25] It is notable that the addition and scalar multiplication of **FN**s (mentioned before) can be reevaluated based on their α-values as follows:

$$(\widetilde{A} \oplus \widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha,$$
$$(\lambda \otimes \widetilde{A})_\alpha = \begin{cases} \lambda \widetilde{A}_\alpha, & \text{if} \quad \lambda > 0, \\ \lambda \widetilde{A}_{1-\alpha}, & \text{if} \quad \lambda < 0. \end{cases}$$

3 Fuzzy Distance Measure

In this section, a fuzzy distance measure between two **FN**s is introduced according to a two-step procedure. In the first step, an exact distance measure is proposed. Using such distance measure, a fuzzy distance measure is proposed. To do these, a popular definition of a (non-fuzzy) distance measure between two **FN**s is first recalled.

Definition 3. A mapping $d : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to [0, \infty)$ is called a (non-fuzzy) distance measure if any $\widetilde{A}, \widetilde{B}$ and $\widetilde{C} \in \mathcal{F}(\mathbb{R})$ meets the following conditions:

1) $d(\widetilde{A}, \widetilde{B}) = 0$ if and only if $\widetilde{A} = \widetilde{B}$,

2) $d(\widetilde{A}, \widetilde{B}) = d(\widetilde{B}, \widetilde{A})$,

3) $d(\widetilde{A}, \widetilde{C}) \leq d(\widetilde{A}, \widetilde{B}) + d(\widetilde{B}, \widetilde{C})$.

Theorem 1. For two **FN**s of $\widetilde{A}$ and $\widetilde{B}$, define

$$d_a(\widetilde{A}, \widetilde{B}) = \int_0^1 g_{\widetilde{A},\widetilde{B}}(\alpha) d\alpha,$$

where

1)

$$g_{\widetilde{A},\widetilde{B}}(\alpha) = \int_{\alpha/2}^{1-\alpha/2} |\widetilde{A}_\beta - \widetilde{B}_\beta| f_{(\alpha/2,0.5,1-\alpha/2)}(\beta) d\beta,$$

2) $f_{\alpha/2,0.5,1-\alpha/2}$ denotes the triangular density function on $[\alpha/2, 1-\alpha/2]$ defined as

$$f_{(\alpha/2,0.5,1-\alpha/2)}(\beta) = \begin{cases} \frac{4(\beta-\alpha/2)}{(1-\alpha)^2}, & \alpha/2 \leq \beta \leq 0.5, \\ \frac{4(1-\alpha/2-\beta)}{(1-\alpha)^2}, & 0.5 \leq \beta \leq 1-\alpha/2. \end{cases} \tag{7}$$

Then, $d_a : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to [0, \infty)$ is a non-fuzzy distance measure.

Proof. To prove, it is enough to show that d_a satisfies assertions (1)-(3) in Definition 4. To check assertion (1), note that $d_a(\widetilde{A}, \widetilde{B}) = 0$ if and only if $g_{\widetilde{A},\widetilde{B}} = 0$ for any $\alpha \in [0, 1]$, i.e. $\widetilde{A}_\beta = \widetilde{B}_\beta$ for any $\beta \in [0, 1]$ which means $\widetilde{A} = \widetilde{B}$. Assertion (2) is also immediately followed. To prove assertion (3), by the conventional triangular inequality, we have

$$|\widetilde{A}_\beta - \widetilde{C}_\beta| \leq |\widetilde{A}_\beta - \widetilde{B}_\beta| + |\widetilde{B}_\beta - \widetilde{C}_\beta|,$$

for any $\beta \in [0, 1]$. Integrating on both sides with respect to $f_{(\alpha/2,0.5,1-\alpha/2)}(\beta)$ on $[\alpha/2, 1-\alpha/2]$, will result in $g_{\widetilde{A},\widetilde{C}}(\alpha) \leq g_{\widetilde{A},\widetilde{B}}(\alpha) + g_{\widetilde{B},\widetilde{C}}(\alpha)$ for any $\alpha \in [0, 1]$. This means $\widetilde{d}_a(\widetilde{A}, \widetilde{C}) \leq (\widetilde{d}_a(\widetilde{A}, \widetilde{B}) + \widetilde{d}_a(\widetilde{B}, \widetilde{C}))$ which completes the proof. □

It is noticeable that $f_{(\alpha/2,0.5,1-\alpha/2)}(\beta)$ cares about points near 0.5 (that is those values near to modal of fuzzy numbers) more than other values in $[\alpha/2, 1-\alpha/2]$. Interestingly, d_a can be interpreted as the double expected value of uniform distribution on $[0,1]$ and a triangular density function on $[\alpha/2, 1-\alpha/2] \subseteq [0,1]$.

Now, a procedure is suggested to extend the non-fuzzy distance measure of d_a as "about d_a". First, a common notion of fuzzy distance measure is reviewed.

Definition 4. [24] We say that $\widetilde{D} : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}([0,\infty))$ is a fuzzy distance measure (**FDM**) if any $\widetilde{A}, \widetilde{B}$ and $\widetilde{C} \in \mathcal{F}(\mathbb{R})$ meet the following conditions:

1) $\widetilde{D}(\widetilde{A}, \widetilde{B}) = I\{0\}$ if and only if $\widetilde{A} = \widetilde{B}$,

2) $\widetilde{D}(\widetilde{A}, \widetilde{B}) = \widetilde{D}(\widetilde{B}, \widetilde{A})$,

3) $\widetilde{D}(\widetilde{A}, \widetilde{C}) \preceq (\widetilde{D}(\widetilde{A}, \widetilde{B}) \oplus \widetilde{D}(\widetilde{B}, \widetilde{C}))$.

Lemma 2. For two **FN**s of $\widetilde{A}$ and $\widetilde{B}$, define

$$(\widetilde{d_a}(\widetilde{A}, \widetilde{B}))_\alpha = \begin{cases} \int_0^1 g_{\widetilde{A},\widetilde{B}}(2w\alpha)dw, & 0 \le \alpha \le 0.5, \\ \int_0^1 g_{\widetilde{A},\widetilde{B}}(1-2(1-w)(1-\alpha))dw, & 0.5 \le \alpha \le 1. \end{cases} \tag{8}$$

Then, $\widetilde{d_a}$ is a **FN**.

Proof. For every $\alpha_1 < \alpha_2$, first, note that $I_{\alpha_2} = [\alpha_2/2, 1-\alpha_2/2] \subseteq I_{\alpha_1} = [\alpha_1/2, 1-\alpha_1/2]$ and $f_{(\alpha_2/2,0.5,1-\alpha_2/2)}(\beta) \le f_{(\alpha_1/2,0.5,1-\alpha_1/2)}(\beta)$ for any $\beta \in [0,1]$. This simply implies that $g_{\widetilde{A},\widetilde{B}}(\alpha_2) \le g_{\widetilde{A},\widetilde{B}}(\alpha_1)$ for every $0 \le \alpha_1 < \alpha_2 \le 1$, i.e. $g_{\widetilde{A},\widetilde{B}}(\alpha)$ is a decreasing function on $[0,1]$. This immediately concludes that $(\widetilde{d_a}(\widetilde{A}, \widetilde{B}))_\alpha$ is also a decreasing function with respect to α. Moreover, it is seen that $(\widetilde{d_a}(\widetilde{A}, \widetilde{B}))_{0.5} = d_a(\widetilde{A}, \widetilde{B})$ is a constant number. These verify that $\widetilde{d_a}$ is a **FN**. □

Theorem 2. Recall assumptions in Lemma 2. Then $\widetilde{d_a} : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}([0,\infty))$ is a **FDM**.

Proof. We show that $\widetilde{d_a}(\widetilde{A}, \widetilde{B})$ meets all the conditions of Definition 4. Future, $\widetilde{d_a}(\widetilde{A}, \widetilde{B}) = I\{0\}$ if and only if for any $\alpha \in [0,1]$, $(\widetilde{d_a}(\widetilde{A}, \widetilde{B}))_\alpha = 0$. This simply concludes that $\widetilde{A}_\alpha = \widetilde{B}_\alpha$ for any $\alpha \in [0,1]$ or $\widetilde{A} = \widetilde{B}$. This shows (1).The assertion (2) can be immediately followed. To prove assertion (3), it is enough to see that $g_{\widetilde{A},\widetilde{C}}(\alpha) \le g_{\widetilde{A},\widetilde{B}}(\alpha) + g_{\widetilde{B},\widetilde{C}}(\alpha)$. This simply implies that $(\widetilde{d_a}(\widetilde{A}, \widetilde{C}))_\alpha \le ((\widetilde{d_a}(\widetilde{A}, \widetilde{B}))_\alpha + (\widetilde{d_a}(\widetilde{B}, \widetilde{C})))_\alpha$ for any $\alpha \in [0,1]$ which completes the proof by Remark 1 and Definition 2. □

Next, the most important properties needed for an absolute error distance are verified in a fuzzy domain.

Lemma 3. If $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \mathcal{F}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, then

1) $\widetilde{d_a}(\widetilde{A} \oplus \widetilde{C}, \widetilde{B} \oplus \widetilde{C}) = \widetilde{d_a}(\widetilde{A}, \widetilde{B})$.

2) $\widetilde{d_a}(\lambda \otimes \widetilde{A}, \lambda \otimes \widetilde{B}) = |\lambda| \otimes \widetilde{d_a}(\widetilde{A}, \widetilde{B})$.

Proof. For three **FN**s of $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$, it is easy to verify that

1) $g_{\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}}(\alpha) = g_{\widetilde{A},\widetilde{B}}(\alpha)$,

2) $g_{\lambda\otimes\widetilde{A},\lambda\otimes\widetilde{B}}(\alpha) = |\lambda| g_{\widetilde{A},\widetilde{B}}(\alpha)$,

for any $\alpha \in [0,1]$. Therefore,

$$(\widetilde{d}_a(\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}))_\alpha = \begin{cases} \int_0^1 g_{\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}}(2w\alpha)dw & 0 \le \alpha \le 0.5, \\ \int_0^1 g_{\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}}(1-\alpha))dw & 0.5 \le \alpha \le 1, \end{cases} \tag{9}$$

$$= \begin{cases} \int_0^1 g_{\widetilde{A},\widetilde{B}}(2w\alpha)dw & 0 \le \alpha \le 0.5, \\ \int_0^1 g_{\widetilde{A},\widetilde{B}}(1-\alpha))dw & 0.5 \le \alpha \le 1, \end{cases} = (\widetilde{d}_a(\widetilde{A},\widetilde{B}))_\alpha, \tag{10}$$

for any $\alpha \in [0,1]$. This concludes that $\widetilde{d}_a(\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}) = \widetilde{d}_a(\widetilde{A},\widetilde{B})$. The second assertion can be also verified due to $g_{\lambda\otimes\widetilde{A},\lambda\otimes\widetilde{B}}(\alpha) = |\lambda| g_{\widetilde{A},\widetilde{B}}(\alpha)$ for any $\alpha \in [0,1]$. □

Remark 2. If two fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$ reduce to non-fuzzy quantities of a and b, then it can be checked that $\widetilde{d}_a(\widetilde{A},\widetilde{B}) = I(|a-b|)$ which is the conventional absolute error distance.

Note that a distance measure should be able to tolerate small errors in evaluating the membership functions that is the distance between two **FN**s should not change if the variation of the membership functions is sufficiently small [**?**]. Here, a notion of robustness for a **FDM** is defined.

Definition 5. Let $\widetilde{D} : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}([0,\infty))$ be a **FDM**. We say $\widetilde{D}$ is robust, if for any given pair of **FN**s $(\widetilde{A},\widetilde{B})$ and a sequence of pair **FN**s of $\{(\widetilde{A}_n,\widetilde{B}_n)\}$ with $d_H(\widetilde{A}_n,\widetilde{A}) \to 0$ and $d_H(\widetilde{B}_n,\widetilde{B}) \to 0$ as $n \to \infty$ we have $d_H(\widetilde{D}(\widetilde{A}_n,\widetilde{B}_n),\widetilde{D}(\widetilde{A},\widetilde{B})) \to 0$ as $n \to \infty$ in which d_H is the Hausdorff distance measure between two **FN**s defined by $d_H(\widetilde{A},\widetilde{B}) = \sup_{\alpha\in[0,1]} |\widetilde{A}_\alpha - \widetilde{B}_\alpha|$.

Now, the robustness of the proposed **FDM** of $\widetilde{d}_a$ is examined by the following theorem.

Theorem 3. The **FDM** of $\widetilde{d}_a$ introduced in Lemma 2 is robust.

Proof. For a given pair of **FN**s $(\widetilde{A},\widetilde{B})$ and a sequence of **FN** pairs $\{(\widetilde{A}_n,\widetilde{B}_n)\}$, assume that $d_H(\widetilde{A}_n,\widetilde{A}) \to 0$ and $d_H(\widetilde{B}_n,\widetilde{B}) \to 0$ as $n \to \infty$. First, for every $\alpha \in [0,1]$, note that

$$(\widetilde{d}_a(\widetilde{A},\widetilde{B}))_\alpha \le \sup_{\alpha\in[0,1]} |\widetilde{A}_\alpha - \widetilde{B}_\alpha| = d_H(\widetilde{A},\widetilde{B}). \tag{11}$$

By Theorem 2 and Equation (11), it follows that

$$\begin{aligned} (\widetilde{d}_a(\widetilde{A}_n,\widetilde{B}_n))_\alpha &\le (\widetilde{d}_a(\widetilde{A}_n,\widetilde{A}))_\alpha + (\widetilde{d}_a(\widetilde{B}_n,\widetilde{A}))_\alpha, \\ &\le (\widetilde{d}_a(\widetilde{A}_n,\widetilde{A}))_\alpha + (\widetilde{d}_a(\widetilde{B}_n,\widetilde{B}))_\alpha + (\widetilde{d}_a(\widetilde{A},\widetilde{B}))_\alpha, \\ &\le d_H(\widetilde{A}_n,\widetilde{A}) + d_H(\widetilde{B}_n,\widetilde{B}) + (\widetilde{d}_a(\widetilde{A},\widetilde{B}))_\alpha^\beta. \end{aligned}$$

Similarly, we have $(\widetilde{d}_a(\widetilde{A},\widetilde{B}))_\alpha \le d_H(\widetilde{A}_n,\widetilde{A}) + d_H(\widetilde{B}_n,\widetilde{B}) + (\widetilde{d}_a(\widetilde{A}_n,\widetilde{B}_n))_\alpha$. This concludes that $|(\widetilde{d}_a(\widetilde{A}_n,\widetilde{B}_n))_\alpha - (\widetilde{d}_a(\widetilde{A},\widetilde{B}))_\alpha| \le d_H(\widetilde{A}_n,\widetilde{A}) + d_H(\widetilde{B}_n,\widetilde{B})$ for every $\alpha \in [0,1]$. Therefore, $d_H(\widetilde{d}_a(\widetilde{A}_n,\widetilde{B}_n),\widetilde{d}_a(\widetilde{A},\widetilde{B})) \to 0$ as $n \to \infty$. This shows that $\widetilde{d}_a$ is robust. □

Example 2. Consider two **TFN**s $\widetilde{A}_1 = (7; 1.3, 0.7)_T$ and $\widetilde{A}_2 = (3; 0.4, 0.2)_T$. First, note that $d_a(\widetilde{A}_1, \widetilde{A}_2) = 3.898$. Furthermore, the **FDM** of $\widetilde{d}_a$ between $\widetilde{A}_1$ and $\widetilde{A}_2$ was evaluated as "about 3.898" whose membership degree is depicted in Figure 1.

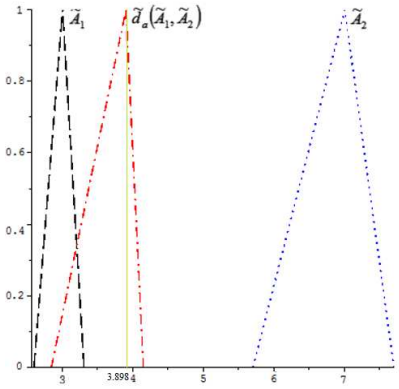

Figure 1: Plot of $\widetilde{d}_a(\widetilde{A}_1, \widetilde{A}_2)$ in Example 2.

Remark 3. Chakraborty and Chakraborty [10] suggested a fuzzy distance measure ($\widetilde{d}^*$) for generalized fuzzy numbers. Ganbari and Nuraei [16] verified that Chakraborty and Chakraborty's distance is not always a generalized triangular fuzzy number. Further, it may be a nonnegative generalized triangular fuzzy number of $\widetilde{d}^*$. Additionally, $\widetilde{d}^*(\widetilde{A} \oplus \widetilde{C}, \widetilde{B} \oplus \widetilde{C}) \neq \widetilde{d}^*(\widetilde{A}, \widetilde{B})$ which is true for $\widetilde{D}_a$ as discussed in Lemma 3. Voxman [41] proposed a notion of fuzzy absolute error distance of $\widetilde{\Delta}$ between two fuzzy numbers. However, the main disadvantage of such distance measure is that $\widetilde{\Delta}(\widetilde{A} \oplus \widetilde{C}, \widetilde{B} \oplus \widetilde{C}) \neq \widetilde{\Delta}(\widetilde{A}, \widetilde{B})$ which is expected for an absolute error distance in a fuzzy environment. Abbasbandy and Hajighasemi [4], introducing a definition for fuzzy distance measure, proposed a symmetric triangular fuzzy distance measure $\widetilde{d}$ which satisfies all conditions in Lemma 3. But, the main shortcoming of $\widetilde{d}$ is that if $\widetilde{d}(\widetilde{A}, \widetilde{B}) = I\{0\}$ then it is not necessary to have $\widetilde{A} = \widetilde{B}$. Sadi-Nezhad et al. [35] proposed a triangular fuzzy absolute error distance measure of $\widetilde{d}$ between two triangular fuzzy numbers of $\widetilde{A} = (a; l_a, r_a)_T$ and $\widetilde{B} = (b; l_b, r_b)_T$. Therefore, the possible applications of a such distance measure are limited in space of **FN**s. Another disadvantage of $\widetilde{d}$ is that $\widetilde{d}(\widetilde{A}, \widetilde{A}) \neq (0; 0, 0)_T$. Beigi et al. [7] proposed a triangular absolute error fuzzy distance measure of $\widetilde{d}$ for two triangular fuzzy numbers of $\widetilde{A} = (a; l_a, r_a)_T$ and $\widetilde{B} = (b; l_b, r_b)_T$. However, it can be observed that $\widetilde{d}(\lambda \otimes \widetilde{A}, \lambda \otimes \widetilde{B}) \neq |\lambda| \otimes \widetilde{d}(\widetilde{A}, \widetilde{B})$ in the case where $\lambda < 0$. Moreover, such a fuzzy distance measure can be only used for triangular **FN**s. Chen and Wang [13] proposed a probability fuzzy absolute error distance measure [12] between two *LR*-fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$. However, in cases where $\widetilde{d}(\widetilde{A}, \widetilde{B}) = I\{0\}$, one can observe that $\widetilde{A} = \widetilde{B}$ is not guaranteed. Moreover, Hesamian and Akbari [24] proposed a fuzzy distance measure between two *LR*-**FN**s based on the absolute value of fuzzy numbers and an extended subtraction operation. The potential advantages of such fuzzy distance lie in its robustness and satisfaction of all conditions in Lemma 3. However, the proposed fuzzy absolute error distance can be used only for *LR*-**FN**s while the proposed method is applicable on any type of **FN**s. Moreover, a simpler procedure to compute the fuzzy distance between two **FN**s is suggested in this paper compared to Hesamian and Akbari's method.

4 Conclusion

This paper proposed a fuzzy distance measure between two fuzzy numbers using α-values as a fuzzy number. The main expected properties of the proposed fuzzy distance measure were also verified in the space of fuzzy numbers. Calculating the fuzzy distance between two fuzzy numbers was demonstrated using a numerical example. The advantages of the proposed fuzzy distance measure were also examined and compared with other existing ones. Future studies should be devoted to extending a notion of distance measure in the space of intuitionistic fuzzy numbers or type-2 fuzzy numbers.

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Modeling and Solving a Multi-objective Location-Routing Problem Considering the Evacuation of Casualties and Homeless People and Fuzzy Paths in Relief Logistics

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Abstract. The relief logistics and humanitarian supply chain in academic literature refer to the process of planning, execution, and effective controlling of the flow of costs and information and storage of necessary goods and materials from the point of origin to consumption with the primary purpose of reducing and relieving the affected people suffer. This paper discusses a multi-objective model for multi-period location-distribution-routing problems considering the evacuation of casualties and homeless people and fuzzy paths in relief logistics. Firstly, an uncertain multi-objective model of the problem was developed based on uncertain parameters of demand, time, and transport capacity, and then, using the fuzzy programming method, uncertain parameters of the problem were controlled. As the problem is NP-hard and GAMS software has not able to solve the model in larger sizes, meta-heuristic algorithms of NSGA-II and MOPSO were used to solve the problem.

Keywords. Relief logistics, Fuzzy programming, Uncertainty, Meta-heuristic algorithm.

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1 Introduction

Natural disasters usually cause serious damage to urban infrastructure and cause severe injury and death in populated areas. In recent years, much attention has been paid to these issues. In the case of natural disasters, rapid distribution of resources is essential to minimize damage and losses. A logistics operation that is carried out to help human beings in crises is called humanitarian logistics. Humanitarian logistics encompasses all processes for estimating, supplying, transporting, storing, and distributing goods, equipment, and services for injured people and relief teams. Logistics in normal circumstances and humanitarian logistics have similarities and differences, such as the similarities of supply chains in the normal and critical condition that the supply chain operates in a critical condition, as in ordinary circumstances, in two ways, forward and backward. In humanitarian logistics, in the forward mode, the goal is: to send and distribute goods and relief items, to send aid forces and medical and medical personnel, and in the backward mode, the collection and burial of the dead, the collection and transfer of injured to local or regional medical centers and the transfer of survivors to the safe (evacuation) areas. What distinguishes humanitarian logistics from ordinary logistics is that under critical conditions, the relief supply chain must act at high speed and aim to preserve human lives. While under ordinary circumstances, the supply chain operates at the lowest cost according to the schedule (Rawls et al., 2010) [17].

2 Literature Review

Emergency logistic operations are generally divided into two phases which are before and after the disaster. Since the focus of this paper is on the post-crisis phase and in the area of short-term and operational planning, an overview of the studies and researches carried out in this field is presented.

The first optimization models in emergency logistics were introduced in the late 1970s after a few marine disasters in the late 1960s and 1970s. Since the 1980s, research on other major disasters (such as storms, floods, and earthquakes) which happen on a large scale has also been included. Oh and Haqqani (1996) in [15] analyzed the transportation of large quantities of commodities such as food, clothing, medical supplies, drugs, machinery, and human resources into an effective approach to minimizing casualties with several types of transportation vehicles for relief operations; And Oudzamar and Laynet (2011) in [16] presented a mathematical model for transporting goods in the response phase, in which the vehicle travel time minima were considered. Bozorgi Amiri et al. (2011) presented a multi-disciplined randomized variable programming model under uncertain conditions [4]. They considered the parameters of demand, supply, and the cost of purchasing and transportation in their proposed model to be uncertain, and used a scenario-based approach. Muralie et al. (2012) considered the problem of the situation - the facility to find out the locations of the city that needs drug-to-be distributed among the population [14]. They considered the identification of the stored facilities and the coverage function intervals to maximize the coverage.

Anne et al. (2013), presented an article entitled "Locating Displacement Transportation Facility under Facility Failure Conditions", to plan the evacuation of many people living at an accidental site [2]. The goal is to minimize the cost of building a facility, the cost of transfer and, the cost of using the facility. Abu Naser et al. (2014) also developed a model for optimizing humanitarian relief centers by considering three goals of minimizing the total time of relief in damaged areas, minimizing the number of employees engaged in these centers, and maximizing the coverage of relief items over the damaged area [1]. This study considered only the definite demands of the damaged areas and only responded with a precise solution (ϵ-constraint) method. Zokaei et al. (2016) considered a three-level supply chain model, including suppliers, relief centers, and damaged districts for uncertain rescue operations and humanitarian relief [21]. Their model seeks to maximize the satisfaction of the affected people while minimizing the costs of the supply chain. Rezaei Malik et al. (2016) designed a two-objective model for handling natural disasters [18]. Their main goal in this article was to achieve optimal planning for degradable commodities such as medical items and milk in central warehouses before the disaster. The objective functions considered for their model included simultaneously minimizing the total operating costs before and after the incident and minimizing the average response time to demand points. In their model, they considered the parameters such as transportation time, demand, reliability and, cost of fines in a non-deterministic way. BozorgiAmiri et al. (2016) considered two issues of the evacuation of the wounded as well as the distribution of relief items simultaneously and used the constant optimization model to answer the problem's uncertainty [5].

In the following, we study the other researches since 2017:

Hu et al. (2017) designed a randomized optimization model for joint inventory decisions before an incident and the transportation of humanitarian relief items after an accident [10]. In their model, they considered the demand parameter as an uncertain parameter and controlled the parameter by probabilistic optimization. Bonomi et al. (2017) reviewed the issues related to locating emergency logistics facilities based on various types of modeling and types of problems before and after the disaster [3]. They examined the problem of locating facilities in four types of definite, dynamic, probabilistic, and stable conditions, and described their solution and locating methods. Torabi et al. (2018) designed a humanitarian supply chain model based on the scenario and under uncertainty [19]. Their goal was to reduce the costs of the entire supply chain network, including the costs of locating, transportation, relief items maintenance, and fines in the case of the supply shortage. In their model, they considered parameters such as fixed costs of construction, transportation costs, and demand to be uncertain and used a scenario-based method. Yahyaei and BozorgiAmiri (2018) started to design a relief chain network under uncertainty [20]. Their main goal in this paper was to control the failure of distribution centers to meet the demand for demand centers as a reduction in investment costs. They also used a robust optimization method to control non-deterministic parameters and showed that total investment costs increase with increasing uncertainty. Alchy and Novian (2018) focused on modeling a scenario-based humanitarian supply chain network [8]. In their model, they looked at the optimal number and location of facilities, facility capacity, inventory levels, and transport net-

work conditions under the uncertainty of demand after the incident. To solve their probable model, they used a shunt-cutting algorithm based on Bandar's breakdown.

This paper discusses a multi-objective model for multi-period location-distribution-routing problems considering the evacuation of casualties and homeless people and fuzzy paths in relief logistics. Some parameters are considered uncertain, including demand, the capacity of vehicles & time. Finally, NSGA-II and MOPSO as the meta-heuristic algorithms have been used to solve the problem in larger sizes.

3 Problem Definition and Modeling

In this paper, a six-level relief logistics issue (inventory of goods, relief distribution centers, damaged areas, temporary shelter sites, temporary medical centers, and hospitals) is considered. Given 1, the problem is considered for post-crisis and pre-crisis situations. Thus, in the pre-crisis conditions, some potential areas for the warehouse of goods, relief distribution centers, temporary accommodation centers, and temporary medical centers are considered, and in post-crisis situations, these centers are quickly located, and the allocation of goods and vehicle routing are dealt with. In this article, the damaged areas have two different types of demand. The first type of request relates to relief supplies and the second type is related to the transfer of survivors to other centers. The survivors in this type of network are divided into three categories. First-class injured people whose condition is critical and immediately transferred to hospitals with relief vehicles such as ambulances or helicopters. Second-class injured patients who have unclear conditions are transported to temporary care centers for treatment. After that, they will be transferred to hospitals if they have a critical condition. Otherwise, they will be transferred to temporary shelter sites. Third-class victims are homeless persons who move to temporary accommodation centers. At this stage, proper routing of vehicles between centers and also the optimal allocation of vehicles is important. On the other hand, the transportation of the injured person is not the only consideration, but it is necessary to send relief items toward affected areas and providing critical items such as water, blankets, and other items for homeless people in temporary accommodation centers. Therefore, the required items for each part are sent by the cargo vehicles from the warehouse to the relief distribution centers, and after the breakdown, relief items are sent to the affected areas while critical items are sent to the temporary accommodation centers. At this stage, the optimal routing of cargo vehicles is another issue. In addition, cargo and relief vehicles, depending on the type of problem and its vital importance, should choose the optimal route of transportation in the shortest possible time to transfer injured persons or emergency aid items. Therefore, in the design of such a network, the transmission time, demand, and capacity of vehicles are considered uncertain.

Given the definition of the stated problem, this is a multi-objective problem that pursues the following opposite objectives:

1. Minimizing total network costs, such as fixed costs of construction, transportation, and inventory

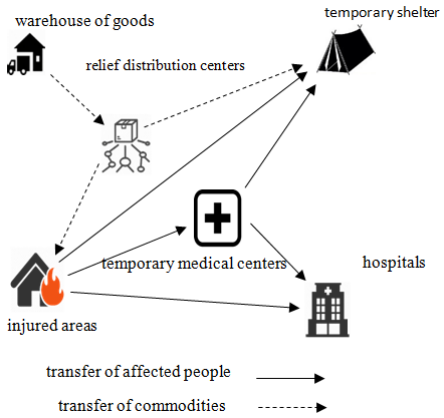

Figure 1: Proposed relief logistics network.

2. Minimize unsatisfied demand for relief items or transferring injured to other centers

3. Minimizing the total number of vehicles

In order to achieve the three above objectives simultaneously, it is important to determine the optimal location of the facility (inventory of goods, relief distribution centers, temporary accommodation centers, and temporary medical centers), optimal allocation of goods to other centers, and optimal routing of cargo and relief vehicles that all form the framework of the problem. With regard to the following assumptions, location-distribution-routing in the relief logistics will be modeled in uncertain conditions:

1. Transportation of survivors and distribution of relief items are considered together.

2. The fleet of vehicles is heterogeneous, and various types of vehicles are available.

3. Simultaneous transportation of goods and survivors in one vehicle is not possible.

4. Categorizing the survivors (1) the injured in a state of emergency; 2) an injured person whose condition is unclear and requires an initial examination; 3) the homeless people.

5. The amount of demand, the number of people in the three categories, the capacity of the vehicles, and the transmission time are uncertain.

6. The capacity of hospitals, temporary accommodation centers, temporary medical centers, relief distribution centers, and warehouses are specific parameters.

7. All centers within the network may receive goods or services from several related facilities.

Given the assumptions and objectives, the problem set, parameters and variables are presented in the next section.

3.1 Location-distribution-routing model in relief logistics

The set, parameters, and decision variables of the basic model are as follows:
Sets:

i : set of affected areas by disaster $i = \{1, 2, \ldots, I\}$,
l : set of potential shelter sites $l = \{1, 2, \ldots, L\}$,
m : set of potential temporary medical centers $m = \{1, 2, \ldots, M\}$,
p : set of potential relief distribution centers $p = \{1, 2, \ldots, P\}$,
h : set of available hospitals $h = \{1, 2, \ldots, H\}$,
g : set of potential warehouses of goods $g = \{1, 2, \ldots, G\}$,
t : set of periods in the time horizon $t = \{1, 2, \ldots, T\}$,
k_1 : set of vehicle types for carrying of relief commodities $k_1 = \{1, 2, \ldots, K_1\}$,
k_2 : set of vehicle types for carrying of people $k_2 = \{1, 2, \ldots, K_2\}$,
k : set of total vehicle types $k = k = K_1 \cup K_2$,
r : set of survivors $r = \{R_A, R_b, R_c\}$,
e_1 : set of relief commodities $e_1 = \{1, 2, \ldots, E_1\}$,
e_2 : set of critical commodities $e_2 = \{1, 2, \ldots, E_2\}$,
e : set of total commodity types $e = e_1 \cup e_2$.

Parameters:

$\widetilde{D1}_{i,r,t}$: number of survivors type waiting at area i at time t,
$\widetilde{D2}_{i,e_1,t}$: amount demanded of commodity type e_1 at area i at time t,
$\widetilde{D3}_{l,e_2,t}$: amount demanded of commodity type e_2 at shelter l at time t,
$\widetilde{Tuh}_{i,h,k_2}$: The estimated time of travel of vehicle type k_2 from affected area i to hospital h,
$\widetilde{Tum}_{i,m,k_2}$: The estimated time of travel of vehicle type k_2 from affected area i to temporary medical center m,
$\widetilde{Tmh}_{m,h,k_2}$: The estimated time of travel of vehicle type k_2 from temporary medical center m to hospital h,
$\widetilde{Tgp}_{g,p,k_1}$: The estimated time of travel of vehicle type k_1 from warehouse g to relief distribution center p,
$\widetilde{Tpi}_{p,i,k_1}$: The estimated time of travel of vehicle type k_1 from relief distribution center p to affected area i,
$\widetilde{Tpl}_{p,l,k_1}$: The estimated time of travel of vehicle type k_1 from relief distribution center p to temporary shelter sites l,
$\widetilde{Cap}_k$: load capacity of vehicle type k,
$Trih_{i,h,k_2}$: The estimated cost of travel of vehicle type k_2 from affected area i to hospital h,
$Trim_{i,m,k_2}$: The estimated cost of travel of vehicle type k_2 from affected area i to temporary medical center m,
$Trmh_{m,h,k-2}$: The estimated cost of travel of vehicle type k_2 from temporary medical center m to hospital h,

$Tril_{i,l,k_2}$: The estimated cost of travel of vehicle type $k-2$ from affected area i to temporary shelter sites l,
$Trml_{m,l,k_2}$: The estimated cost of travel of vehicle type k_2 from temporary medical center m to temporary shelter sites l,
$Trgp_{g,p,k_1}$: The estimated cost of travel of vehicle type k_1 from warehouse g to relief distribution center p,
$Tripi_{p,ik_1}$: The estimated cost of travel of vehicle type k_1 from relief distribution center p to affected area i,
$Trpl_{p,l,k_1}$: The estimated cost of travel of vehicle type k_1 from relief distribution center p to temporary shelter site l,
$CapL_{l,t}$: The capacity of temporary shelter site l for affected people type C at time t,
$CapM_{m,t}$: The capacity of temporary medical center m for affected people type B at time t,
$CapH_{h,t}$: The capacity of hospital h for affected people type A at time t,
$CapG_{g,e,t}$: The storage capacity of the commodity e at warehouse g at time t,
$CapP_{p,e,t}$: The storage capacity of the commodity e at the relief distribution center p at time t,
$Hd_{e_1,t}$: Inventory cost of commodity e_1 at time t,
$FixL_l$: Fixed cost for opening a new shelter center l,
$FixM_m$: Fixed cost for opening a new temporary medical center m,
$FixG_g$: Fixed cost for opening a new warehouse g,
$FixP_p$: Fixed cost for opening a new relief distribution center p,
$FixK_k$: Fixed cost for vehicle type k,
$NT_{k,t}$: Maximum number of vehicles type k available at time t,
F_{m,R_b} : percentage of affected people type B that transported to temporary shelter sites after cure at time t,
$BigM$: a big number,
$Mtime1$: Maximum time for transporting of injured people between network levels,
$Ntime2$: Maximum time for transporting of commodities between network levels,
ω : Normalization weight for second objective function.

Decision variables:

X_{i,h,r_a,t,k_2} : Number of survivors type A transported from affected area i to hospital h by,
Y_{i,m,r_b,t,k_2} : Number of survivorstype B transported from affected area i to temporary medical,
Z_{i,m,r_b,t,k_2} : Number of survivorstype C transported from affected area i to temporary shelter l,
U_{m,l,r_c,t,k_2} : Number of survivors type C transported from temporary medical center m to,
W_{m,h,r_a,t,k_2} : Number of survivorstype A transported from temporary medical center m to ,
O_{p,i,e_1,t,k_1} : Number of commodity type e_1 transported from relief distribution p to affected,
Q_{p,l,e_2,t,k_1} : Number of commodity type e_2 transported from relief distribution p to temporary,
N_{g,p,e,t,k_1} : Number of commodity type e transported from warehouse$g$ to relief distribution,
X'_{i,h,t,k_2} : $=1$, whether vehicle type k_2 travels across the route i to h at time, $t=0$, otherwise.
Y'_{i,m,t,k_2} : $=1$, whether vehicle type k_2 travels across the route i to m at time,

$t = 0$, otherwise.

Z'_{i,l,t,k_2}: $= 1$, whether vehicle type k_2 travels across the route i to l at time, $t = 0$, otherwise.

W'_{m,h,t,k_2}: $= 1$, whether vehicle type k_2 travels across the route m to h at time, $t = 0$, otherwise.

U'_{m,l,t,k_2}: $= 1$, whether vehicle type k_2 travels across the route m to l at time, $t = 0$, otherwise.

O'_{p,i,t,k_1}: $= 1$, whether vehicle type k_1 travels across the route p to i at time, $t = 0$, otherwise.

Q'_{p,l,t,k_1}: $= 1$, whether vehicle type k_1 travels across the route p to l at time, $t = 0$, otherwise.

N'_{g,p,t,k_1}: $= 1$, whether vehicle type k_1 travels across the route g to p at time, $t = 0$, otherwise.

ZM_m: $= 1$, whether a temporary medical center at m is open, $= 0$, otherwise.

ZL_l: $= 1$, whether a temporary shelter at l is open $= 0$, otherwise.

ZP_p: $= 1$, whether a relief distribution center at p is open, $= 0$, otherwise.

ZG_g: $= 1$, whether a warehouse at g is open, $= 0$, otherwise.

$No_{k,t}$: Number of used vehicles k at time t,

$In_{e_1,t}$: Amount of stored inventory of commodity type e_1 at time t,

$S1_{i,r,t}$: Number of unserved people type r in affected i area at time t,

$S2_{i,e_1,t}$: Amount of unsatisfied demand of commodity type e_1 in affected area i at time t.

Modeling:

$$
\begin{aligned}
\min Z_1 = & \sum_{m} FixM_m \cdot ZM_m + \sum_{l} FixL_l \cdot ZL_l + \sum_{p} FixP_p \cdot ZP_p + \sum_{g} FixG_g \cdot ZG_g \\
& + \sum_{i,h,k_2,t} FixK_{k_2} \cdot X'_{i,h,t,k_2} + \sum_{i,m,k_2,t} FixK_{k_2} \cdot Y'_{i,m,t,k_2} \sum_{i,l,k_2,t} FixK_{k_2} \cdot Z'_{i,l,t,k_2} \\
& + \sum_{l,m,k_2,t} FixK_{k_2} \cdot U'_{m,l,t,k_2} + \sum_{m,h,k_2,t} FixK_{k_2} \cdot W'_{m,h,t,k_2} + \sum_{g,p,k_1,t} FixK_{k_1} \cdot N'_{g,p,t,k_1} \\
& + \sum_{p,i,k_1,t} FixK_{k_1} \cdot O'_{p,i,t,k_1} + \sum_{p,l,k_1,t} FixK_{k_1} \cdot Q'_{p,l,t,k_1} + \sum_{i,h,r_a,k_2,t} Trih_{i,h,k_2} \cdot X_{i,h,r_a,t,k_2} \\
& + \sum_{i,m,r_b,k_2,t} Trim_{i,m,k_2} \cdot Y_{i,m,r_b,t,k_2} + \sum_{i,l,r_c,k_2,t} Tril_{i,c,k_2} \cdot Z_{i,l,r_c,t,k_2} \\
& + \sum_{m,l,r_c,k_2,t} Triml_{m,l,k_2} \cdot U_{m,l,r_c,t,k_2} + \sum_{m,h,r_a,k_2,t} Trmh_{m,h,k_2} \cdot W_{m,h,r_a,t,k_2} \\
& + \sum_{g,p,e,k_1,t} Trgp_{g,p,k_1} \cdot N_{g,p,e,t,k_1} + \sum_{p,i,e_1,k_1,t} Tripi_{p,i,k-1} \cdot O_{p,i,e_1,t,k_1} \\
& + \sum_{p,l,e_2,k_1,t} Trpl_{p,l,k_1} \cdot Q_{p,l,e_2,t,k_1} \sum_{e_1,t} Hd_{e_1,t} \cdot In_{e_1,t}
\end{aligned} \tag{1}
$$

$$\min Z_2 = \omega \sum_{i,r,t} S1_{i,r,t} + (1-\omega) \sum_{i,e_1,t} S2_{i,e_1,t} \tag{2}$$

$$\min Z_3 = \sum_{k,t} No_{k,t} \tag{3}$$

$s.t:$

$$\sum_{h,k_2} X_{i,h,r_a,t,k_2} + \sum_{m,k_2} Y_{i,m,r_b,t,k_2} + \sum_{l,k_2} Z_{i,l,r_c,t,k_2} + \sum_{r} S1_{i,r,t} = \sum_{r} \tilde{D}_{\iota,r,t}, \qquad \forall i,t, \tag{4}$$

$$\sum_{p,k_1} O_{p,i,e_1,t,k_1} - In_{e_1,t} + In_{e_1,t-1} + S2_{i,e_1,t} = \widetilde{D2}_{i,e_1,t}, \qquad \forall i,e_1,t, \tag{5}$$

$$\sum_{p,k_1} Q_{p,l,e_2,t,k_1} = \widetilde{D3}_{l,e_2,t} \cdot ZL_l, \qquad \forall i,e_2,t, \tag{6}$$

$$\sum_{i,k_2} F_{m,R_b} \cdot Y_{i,m,r_b,t,k_2} = \sum_{l,k_2} U_{m,l,r_c,t,k_2}, \qquad \forall m,r,t, \tag{7}$$

$$\sum_{i,k_2} (1 - F_{m,R_b}) \cdot Y_{i,m,r_b,t,k_2} = \sum_{h,k_2} W_{m,h,r_a,t,k_2}, \qquad \forall m,r,t, \tag{8}$$

$$\sum_{g,k_1} N_{g,p,e,t,k_1} = \sum_{i,k_1} O_{p,i,e_1,t,k_1} + \sum_{l,k_1} Q_{p,l,e_2,t,k_1}, \qquad \forall p,e,t, \tag{9}$$

$$\sum_{r_c,i,k_2} Z_{i,l,r_c,t,k_2} + \sum_{r_c,m,k_2} U_{m,l,r_c,t,k_2} \le CapL_{l,t} \cdot ZL_l, \qquad \forall l,t, \tag{10}$$

$$\sum_{r_b,i,k_2} Y_{i,m,r_b,t,k_2} \le CapM_{m,t} \cdot ZM_m, \qquad \forall m,t, \tag{11}$$

$$\sum_{r_a,i,k_2} X_{i,h,r_a,t,k_2} + \sum_{r_a,m,k_2} W_{m,h,r_a,t,k_2} \le CapH_{h,t}, \qquad \forall h,t, \tag{12}$$

$$\sum_{g,k_1} N_{g,p,e,t,k_1} \le CapP_{p,e,t} \cdot ZP_p, \qquad \forall p,e,t, \tag{13}$$

$$\sum_{p,k_1} N_{g,p,e,t,k_1} \le CapG_{g,e,t} \cdot ZG_g, \qquad \forall g,e,t, \tag{14}$$

$$\sum_{r_c,i,l} Z_{i,l,r_c,t,k_2} + \sum_{r_c,m,l} U_{m,l,r_c,t,k_2} + \sum_{r_b,i,m} Y_{i,m,r_b,t,k_2} + \sum_{r_a,i,h} X_{i,h,r_a,t,k_2} + \sum_{r_a,m,h} W_{m,h,r_a,t,k_2} \le \widetilde{Cap}_{k_2} \cdot No_{k_2,t}, \qquad \forall k_2,t, \tag{15}$$

$$\sum_{e,g,p} N_{g,p,e,t,k_1} + \sum_{i,p,e_1} O_{p,i,e_1,t,k_1} + \sum_{p,l,e_2} Q_{p,l,e_2,t,k_1} \le \widetilde{Cap}_{k_1} \cdot No_{k_1,t}, \qquad \forall k_1,t, \tag{16}$$

$$No_{k,t} \le NT_{k,t}, \qquad \forall k,t, \tag{17}$$

$$\sum_{r_c} Z_{i,l,r_c,t,k_2} \le BigM \cdot Z'_{i,l,t,k_2}, \qquad \forall i,l,t,k_2, \tag{18}$$

$$\sum_{r_c} U_{m,l,r_c,t,k_2} \le BigM \cdot U'_{m,l,t,k_2}, \qquad \forall m,l,t,k_2, \tag{19}$$

$$\sum_{r_b} Y_{i,m,r_b,t,k_2} \le BigM \cdot Y'_{i,m,t,k_2}, \forall i,m,t,k_2, \tag{20}$$

$$\sum_{r_a} X_{i,h,r_a,t,k_2} \le BigM \cdot X'_{i,h,t,k_2}, \qquad \forall i,h,t,k_2, \tag{21}$$

$$\sum_{r_a} W_{m,h,r_a,t,k_2} \le BigM \cdot W'_{m,h,t,k_2}, \qquad \forall m,h,t,k_2, \tag{22}$$

$$\sum_{e} N_{g,p,e,t,k_1} \le BigM \cdot N'_{g,p,t,k_1}, \qquad \forall g,p,t,k_1, \tag{23}$$

$$\sum_{e_1} O_{p,i,r_1,t,k_1} \le BigM \cdot O'_{p,i,t,k_1}, \qquad \forall p,i,t,k_1, \tag{24}$$

$$\sum_{e_2} Q_{p,l,r_1,t,k_1} \le BigM \cdot Q'_{p,l,t,k_1}, \qquad \forall p,l,t,k_1, \tag{25}$$

$$\widetilde{Tuh}_{i,h,k_2} \cdot X'_{i,h,t,k_2} \le Mtime1, \qquad \forall i,h,t,k_2, \tag{26}$$

$$\widetilde{Tum}_{i,m,k_2} \cdot Y'_{i,m,t,k_2} \le Mtime1, \qquad \forall i,m,t,k_2, \tag{27}$$

$$\widetilde{Timh}_{m,h,k_2} \cdot W'_{m,h,t,k_2} \le Mtime1, \qquad \forall m,h,t,k_2, \tag{28}$$

$$\widetilde{Tigp}_{g,p,k_1} \cdot N'_{g,p,t,k_1} \le Mtime2, \qquad \forall g,p,t,k_1, \tag{29}$$

$$\widetilde{Tipi}_{p,i,k_1} \cdot O'_{p,i,t,k_1} \le Mtime2, \qquad \forall p,i,t,k_1, \tag{30}$$

$$\widetilde{Tipl}_{p,l,k_1} \cdot Q'_{p,l,t,k_1} \le Mtime2, \qquad \forall p,l,t,k_1, \tag{31}$$

$$No_{k,t} \ge 0, \quad integer \tag{32}$$

$$X_{i,h,r_a,t,k_2}, Y_{i,m,r_b,t,k_2}, Z_{i,m,r_b,t,k_2}, U_{m,l,r_c,t,k_2}, W_{m,h,r_a,t,k_2},$$
$$O_{p,i,e_1,t,k_1}, Q_{p,l,e_2,t,k_1}, N_{g,p,e,t,k_1}, In_{e_1,t}, S1_{i,r,t}, S2_{i,e_1,t} \ge 0, \tag{33}$$

$$X'_{i,h,t,k_2}, Y'_{i,m,t,k_2}, Z'_{i,l,t,k_2}, W'_{m,h,t,k_2}, U'_{m,l,t,k_2}, O'_{p,i,t,k_1},$$
$$Q'_{p,l,t,k_1}, N'_{g,p,t,k_1}, ZM_m, ZL_l, ZP_p, ZG_g \in \{0,1\}. \tag{34}$$

Equation (1) shows the objective function of the problem and involves minimizing the total cost of the relief logistics network. Equation (2) is to minimize the missed estimation of the different items demand in damaged areas and the not-transferred injured people to other centers. Since the variables of the second objective function are not from the same domain, a weighted formula is used. Equation (3) minimizes the total number of cargo and relief vehicles at all times. Equation (4) shows the number of first, second, and third-class injured people transported by relief vehicles to hospitals, temporary medical centers, and temporary shelters. Equation (5) shows the amount of sent relief items from distribution centers to injured areas, along with its balance inventory. Equation (6) shows the rate of transferred critical items from distribution centers to temporary shelters. Equation (7) shows the number of second-class injured patients being treated in temporary medical centers and transferred to temporary shelters. Equation (8) shows the number of second-class injured patients who have not been treated in temporary medical centers but transferred to the hospital. Equation (9) shows the number of relief items transferred from the inventory to distribution centers and how they are distributed. Equations (10) to (14), respectively, show the restrictions related to the capacity of temporary shelters, temporary medical centers, hospitals, relief distribution centers, and supply centers after an incident. Equations (15) and (16) categorize the capacity of various cargo and relief vehicles into the required number of vehicles. Equation (17) ensures that the required number of cargo and relief vehicles do not exceed the number of available vehicles. Equations (18) to (25) are about existing limits related to cargo and relief vehicle routing among the

relief logistics network. Equations (26) to (31) ensure that all vehicles assigned to each center transfer injured people and relief items to other centers within the allocated time. Equations (32) to (34) show the type and domain of the decision variables.

3.2 Uncertain parameters control

To make it functional, computational, and easy in computing, a triangular distribution is used to specify each fuzzy parameter. The distribution can be expressed as the degree of occurrence possibility for an event with uncertain characteristics. Figure 2 shows the distribution of the fuzzy parameter (C^p, C^m, C^0). C^0, C^m and C^p respectively represent the optimistic value, probable value, pessimistic value of fuzzy number $\tilde{C}$, which are determined by the decision-maker.

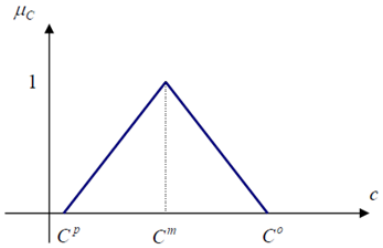

Figure 2: The triangular distribution of the fuzzy parameter $\tilde{C}$.

It should be noted that the probability distribution of fuzzy parameters is determined based on historical records, decision-maker subjective data knowledge. To defuzzify parameter $\tilde{C}$, Dotoli et al. (2017) method has been used [7]. Therefore, the defuzzified model of parameter $\tilde{C}$ will be as Equation (35).

$$C = w_1 C_\alpha^p + w_2 C_\alpha^m + w_3 C_\alpha^o. \tag{35}$$

In above equation

$$\begin{aligned} C_\alpha^p &= C^m + (1-\alpha)(C^p - C^m), \\ C_\alpha^m &= C^m, \\ C_\alpha^o &= C^o + \alpha(C^m - C^o). \end{aligned} \tag{36}$$

In Equation (36), α is the uncertainty rate (fit of alpha) and the values of $w_1 = w_3 = 1/6$ and $w_2 = 4/6$ are in accordance with (Liang, 2011) [12]. Therefore, the ultimate model is as follows. By considering the following uncertain parameters, the certain controlled model can be expressed by the fuzzy programming method as:

$$\begin{aligned} \widetilde{D1}_{i,r,t} &= (D1^p_{i,r,t}, D1^m_{i,r,t}, D1^o_{i,r,t}), \\ \widetilde{D2}_{i,e_1,t} &= (D2^p_{i,e_1,t}, D2^m_{i,e_1,t}, D2^o_{i,e_1,t}), \\ \widetilde{D3}_{l,e_2,t} &= (D3^p_{l,e_2,t}, D3^o_{l,e_2,t}). \\ \widetilde{Tuh}_{i,h,k_2} &= (Tiih^p_{i,h,k_2}, Tiih^m_{i,h,k_2}, Tiih^o_{i,h,k_2}), \end{aligned}$$

$$\widetilde{Tiim}_{i,m,k_2} = (Tiim^{p}_{i,m,k_2}, Tiim^{m}_{i,m,k_2}, Tiim^{o}_{i,m,k_2}),$$
$$\widetilde{Timh}_{m,h,k_2} = (Timh^{p}_{m,h,k_2}, Timh^{m}_{m,h,k_2}, Timh^{o}_{m,h,k_2}),$$
$$\widetilde{Tigp}_{g,p,k_1} = (Tigp^{p}_{g,p,k_1}, Tigp^{m}_{g,p,k_1}, Tigp^{o}_{g,p,k_1}),$$
$$\widetilde{Tipi}_{p,i,k_1} = (Tipi^{p}_{p,i,k_1}, Tipi^{m}_{p,i,k_1}, Tipi^{o}_{p,i,k_1}),$$
$$\widetilde{Tipl}_{p,l,k_1} = (Tipl^{p}_{p,l,k_1}, Tipl^{m}_{p,l,k_1}, Tipl^{o}_{p,l,k_1}),$$
$$\widetilde{Cap}_{k} = (Cap^{p}_{k}, Cap^{m}_{k}, Cap^{o}_{k}).$$

The certain supply chain model is expressed as follows:

$$\begin{aligned}
\min Z_1 = {} & \sum_{m} FixM_m \cdot ZM_m + \sum_{l} FixL_l \cdot ZL_l + \sum_{p} Fix\ P_p \cdot ZP_p + \sum_{g} FixG_g \cdot ZG_g \\
& + \sum_{i,h,k_2,t} FixK_{k_2} \cdot X'_{i,h,t,k_2} + \sum_{i,m,k_2,t} FixK_{k_2} \cdot Y'_{i,m,t,k_2} \sum_{i,l,k_2,t} FixK_{k_2} \cdot Z'_{i,l,t,k_2} \\
& + \sum_{l,m,k_2,t} FixK_{k_2} \cdot U'_{m,l,t,k_2} + \sum_{m,h,k_2,t} FixK_{k_2} \cdot W'_{m,h,t,k_2} + \sum_{g,p,k_1,t} FixK_{k_1} \cdot N'_{g,p,t,k_1} \\
& + \sum_{p,i,k_1,t} FixK_{k_1} \cdot O'_{p,i,t,k_1} + \sum_{p,l,k_1,t} FixK_{k_1} \cdot Q'_{p,l,t,k_1} + \sum_{i,h,r_a,k_2,t} Trih_{i,h,k_2} \cdot X_{i,h,r_a,t,k_2} \\
& + \sum_{i,m,r_b,k_2,t} Trim_{i,m,k_2} \cdot Y_{i,m,r_b,t,k_2} + \sum_{i,l,r_c,k_2,t} Tril_{i,c,k_2} \cdot Z_{i,l,r_c,t,k_2} \qquad (37) \\
& + \sum_{m,l,r_c,k_2,t} Trml_{m,l,k_2} \cdot U_{m,l,r_c,t,k_2} + \sum_{m,h,r_a,k_2,t} Trmh_{m,h,k_2} \cdot W_{m,h,r_a,t,k_2} \\
& + \sum_{g,p,e,k_1,t} Trgp_{g,p,k_1} \cdot N_{g,p,e,t,k_1} + \sum_{p,i,e_1,k_1,t} Trpi_{p,i,k_1} O_{p,i,e_1,t,k_1} \\
& + \sum_{p,l,e_2,k_1,t} Trpl_{p,l,k_1} \cdot Q_{p,l,e_2,t,k_1} \sum_{e_1,t} Hd_{e_1,t} \cdot In_{e_1,t}
\end{aligned}$$

$$\min Z_2 = \omega \sum_{i,r,t} S1_{i,r,t} + (1-\omega) \sum_{i,e_1,t} S2_{i,e_1,t} \qquad (38)$$

$$\min Z_3 = \sum_{k,t} No_{k,t} \qquad (39)$$

s.t

$$\sum_{h,k_2} X_{i,h,r_a,t,k_2} + \sum_{m,k_2} Y_{i,m,r_b,t,k_2} + \sum_{l,k_2} Z_{i,l,r_c,t,k_2} + \sum_{r} S1_{i,r,t} = \sum_{r} \begin{pmatrix} w_1 D1^{\alpha,p}_{i,r,t} + w_2 D1^{\alpha,m}_{i,r,t} \\ +w_1 D1^{\alpha,t}_{i,r,t} \end{pmatrix}, \ \forall i,t, \qquad (40)$$

$$\sum_{p,k_1} O_{p,i,e_1,t,k_1} - In_{e_1,t} + In_{e_1,t-1} + S2_{i,e_1,t} = \begin{pmatrix} w_1 D2^{\alpha,p}_{i,e_1,t} + w_2 D2^{\alpha,m}_{i,e_1,t} \\ +w_3 D2^{\alpha,O}_{i,e_1,t} \end{pmatrix}, \ \forall i,e_1,t, \qquad (41)$$

$$\sum_{p,k_1} Q_{p,l,e_2,t,k_1} = \left(w_1 D3^{\alpha,p}_{l,e_2,t} + w_2 D3^{\alpha,m}_{l,e_2,t} + w_3 D3^{\alpha,O}_{l,e_2,t} \right), \ \forall i,e_2,t, \qquad (42)$$

$$\begin{aligned}
& \sum_{r_c,i,l} Z_{i,l,r_c,t,k_2} + \sum_{r_c,m,l} U_{m,l,r_c,t,k_2} + \sum_{r_b,i,m} Y_{i,m,r_b,t,k_2} + \sum_{r_a,i,h} X_{i,h,r_a,t,k_2} \\
& + \sum_{r_a,m,h} W_{m,h,r_a,t,k_2} \le \begin{pmatrix} w_1 CapK^{\alpha,p}_{k_2} + w_2 CapK^{\alpha,m}_{k_2} \\ +w_3 CapK^{\alpha,O}_{k_2} \end{pmatrix} \cdot No_{k_2,t}, \ \forall k_2,t, \qquad (43) \\
& \sum_{e,g,p} N_{g,p,e,t,k_1} + \sum_{i,p,e_1} O_{p,i,e_1,t,k_1} + \sum_{p,l,e_2} Q_{p,l,e_2,t,k_1} \le
\end{aligned}$$

$$(w_1 CapK_{k_1}^{\alpha,p} + w_2 CapK_{k_1}^{\alpha,m} + w_3 CapK_{k_1}^{\alpha,o}) \cdot No_{k_1,t},\ \forall k_1, t, \tag{44}$$

$$\left(w_1 Tiih_{i,h,k_2}^{\alpha,p} + w_2 Tiih_{i,h,k_2}^{\alpha,m} + w_3 Tiih_{i,h,k_2}\right) \cdot X'_{i,h,t,k_2} \le Mtime1,\ \forall i, h, t, k_2, \tag{45}$$

$$\left(w_1 Tiih_{i,m,k_2}^{\alpha,p} + w_2 Tiim_{i,m,k_2}^{\alpha,m} + w_3 Tiim_{i,m,k_2}\right) \cdot Y'_{i,m,t,k_2} \le Mtime1,\ \forall i, m, t, k_2, \tag{46}$$

$$\left(w_1 Timh_{m,h,k_2}^{\alpha,p} + w_2 Timh_{m,h,k_2}^{\alpha,m} + w_3 Timh_{m,h,k_2}\right) \cdot W'_{m,h,t,k_2} \le Mtime1,\ \forall m, h, t, k_2, \tag{47}$$

$$\left(w_1 Tigp_{g,p,k_1}^{\alpha,p} + w_2 Tigp_{g,p,k_1}^{\alpha,m} + w_3 Tigp_{g,pk_1}^{\alpha,o}\right) \cdot N'_{g,p,tk_1} \le Mtime2,\ \forall g, p, t, k_1, \tag{48}$$

$$\left(w_1 Tipi_{p,i,k_1}^{\alpha,p} + w_2 Tipi_{p,i,k_1}^{\alpha,m} + w_3 Tipi_{p,i,k_1}^{\alpha,o}\right) \cdot O'_{p,i,t,k_1} \le Mtime2,\ \forall p, i, t, k_1, \tag{49}$$

$$\left(w_1 Tipl_{p,l,k_1}^{\alpha,p} + w_2 Tipl_{p,l,k_1}^{\alpha,m} + w_3 Tipl_{p,l,k_1}^{\alpha,o}\right) \cdot Q'_{p,l,t,k_1} \le Mtime2,\ \forall p, l, t, k_1, \tag{50}$$

$$\begin{cases} D1_{i,r,t}^{\alpha,p} = D1_{i,r,t}^{m} + (1-\alpha)(D1_{i,r,t}^{p} - D1_{i,r,t}^{m}) \\ D1_{i,r,t}^{\alpha,m} = D1_{i,r,t}^{m}, \qquad \forall i, r, t \\ D1_{i,r,t}^{\alpha,o} = D1_{i,r,t}^{o} + (\alpha)(D1_{i,r,t}^{m} - D1_{i,r,t}^{o}) \end{cases} \tag{51}$$

$$\begin{cases} D2_{i,e_1,t}^{\alpha,p} = D2_{i,e_1,t}^{m} + (1-\alpha)(D2_{i,e_1,t}^{p} - D2_{i,e_1,t}^{m}) \\ D2_{i,e_1,t}^{\alpha,m} = D2_{i,e_1,t}^{m}, \qquad \forall i, e_1, t \\ D2_{i,e_1,t}^{\alpha,o} = D2_{i,e_1,t}^{o} + (\alpha)(D2_{i,e_1,t}^{m} - D2_{i,e_1,t}^{o}) \end{cases} \tag{52}$$

$$\begin{cases} D3_{l,e_2,t}^{\alpha,p} = D3_{l,e_2,t}^{m} + (1-\alpha)(D3_{i,e_2,t}^{p} - D3_{l,e_2,t}^{m}) \\ D3_{l,e_2,t}^{\alpha,m} = D3_{l,e_2,t}^{m}, \qquad \forall l, e_2, t^m \\ D3_{l,e_2,t}^{\alpha,o} = D3_{l,e_2,t}^{o} + (\alpha)(D3_{i,e_2,t}^{m} - D3_{l,e_2,t}^{o}) \end{cases} \tag{53}$$

$$\begin{cases} CapK_{k}^{\alpha,p} = CapK_{k}^{m} + (1-\alpha)(CapK_{k}^{p} - CapK_{k}^{m}) \\ CapK_{k}^{\alpha,m} = CapK_{k}^{m}, \qquad \forall k \\ CapK_{k}^{\alpha,o} = CapK_{k}^{o} + (\alpha)(CapK_{k}^{m} - CapK_{k}^{o}) \end{cases} \tag{54}$$

$$\begin{cases} Tiih_{i,h,k_2}^{\alpha,p} = Tiih_{i,h,k_2}^{m} + (1-\alpha)(Tiih_{i,h,k_2}^{p} - Tiih_{i,h,k_2}^{m}) \\ Tiih_{i,h,k_2}^{\alpha,m} = Tiih_{i,h,k_2}^{m}, \qquad \forall i, h, k_2 \\ Tiih_{i,h,k_2}^{\alpha,o} = Tiih_{i,h,k_2}^{o} + (\alpha)(Tiih_{i,h,k_2}^{m} - Tiih_{i,h,k_2}^{o}) \end{cases} \tag{55}$$

$$\begin{cases} Tiim_{i,m,k_2}^{\alpha,p} = Tiim_{i,m,k_2}^{m} + (1-\alpha)(Tiim_{i,m,k_2}^{p} - Tiim_{i,m,k_2}^{m}) \\ Tiim_{i,m,k_2}^{\alpha,m} = Tiim_{i,m,k_2}^{m}, \qquad \forall i, m, k_2 \\ Tiim_{i,m,k_2}^{\alpha,o} = Tiim_{i,m,k_2}^{o} + (\alpha)(Tiim_{i,m,k_2}^{m} - Tiim_{i,m,k_2}^{o}) \end{cases} \tag{56}$$

$$\begin{cases} Timh_{m,h,k_2}^{\alpha,p} = Timh_{m,h,k_2}^{m} + (1-\alpha)(Timh_{m,h,k_2}^{p} - Timh_{m,h,k_2}^{m}) \\ Timh_{m,h,k_2}^{\alpha,m} = Timh_{m,h,k_2}^{m}, \qquad \forall m, h, k_2 \\ Timh_{m,h,k_2}^{\alpha,o} = Timh_{m,h,k_2}^{o} + (\alpha)(Timh_{m,h,k_2}^{m} - Timh_{m,h,k_2}^{o}) \end{cases} \tag{57}$$

$$\begin{cases} Tigp_{g,p,k_1}^{\alpha,p} = Tigp_{g,p,k_1}^{m} + (1-\alpha)(Tigp_{g,p,k_1}^{p} - Tigp_{g,p,k_1}^{m}) \\ Tigp_{g,p,k_1}^{\alpha,m} = Tigp_{g,p,k_1}^{m}, \qquad \forall g, p, k_1 \\ Tigp_{g,p,k_1}^{\alpha,o} = Tigp_{g,p,k_1}^{o} + (\alpha)(Tigp_{g,p,k_1}^{m} - Tigp_{g,p,k_1}^{o}) \end{cases} \tag{58}$$

$$\begin{cases} Tipi_{p,i,k_1}^{\alpha,p} = Tipi_{p,i,k_1}^{m} + (1-\alpha)(Tipi_{p,i,k_1}^{p} - Tipi_{p,i,k_1}^{m}) \\ Tipi_{p,i,k_1}^{\alpha,m} = Tipi_{p,i,k_1}^{m}, \qquad \forall g, p, k_1 \\ Tipi_{p,i,k_1}^{\alpha,o} = Tipi_{p,i,k_1}^{o} + (\alpha)(Tipi_{p,i,k_1}^{m} - Tipi_{p,i,k_1}^{o}) \end{cases} \tag{59}$$

$$\begin{cases} Tipl_{p,l,k_1}^{\alpha,p} = Tipl_{p,l,k_1}^{m} + (1-\alpha)(Tipl_{p,l,k_1}^{p} - Tipl_{p,l,k_1}^{m}) \\ Tipl_{p,l,k_1}^{\alpha,m} = Tipl_{p,l,k_1}^{m}, \qquad \forall p, l, k_1 \\ Tipl_{p,l,k_1}^{\alpha,o} = Tipl_{p,l,k_1}^{o} + (\alpha)(Tipl_{p,l,k_1}^{m} - Tipl_{p,l,k_1}^{o}) \end{cases} \tag{60}$$

$$(7)-(14), (17)-(25), (32)-(34) \tag{61}$$

4 Solving Methods

In this paper, due to the NP-hardness of the problem, the NSGA-II and MOPSO algorithms are used to solve the relief logistics problem. Therefore, in this section, the basic principles of the mentioned algorithms are discussed. In the end, we introduce indexes to compare the efficient responses of each algorithm.

4.1 Non-dominated sorting genetic algorithm - II

This algorithm, like the genetic algorithm, begins with a randomly generated primitive population. In the next step, the generated population is evaluated from the viewpoint of the defined objective functions (suppose we have two minimization goal functions). After dividing the population into different categories using the non-dominated sorting process, we calculate the control parameter called the crowding distance. This parameter is calculated for each of two members in each group and represents a measure of the proximity of the target member to the other members of that group. A large amount of this parameter will lead to divergence and a wider range of population members. On the other hand, in this algorithm, among the answers of each generation P_t, some of them are selected using the binary tournament selection method. In the binary selection method, two random responses are selected from the population, and then a comparison is made between the two answers, so the best one is eventually selected. The selection criteria in NSGA-II are primarily the response rank and, secondly, the crowding distance which is related to the answer. The lowest response rank and the highest crowding distance are preferred. By repeating the binary selection on the population of each generation, a set of individuals of that generation is selected to participate in the combination and mutation. The combination function is performed on the part of the selected individuals, and the mutation function is carried out on the rest. As a result, the population Q_t is made up of children and mutated individuals. Subsequently, this population is merged with the main population. The members of the newly formed population R_t are sorted based on their rank in ascending order. Members of the same ranked, are sorted based on crowding distance in descending order. At present, population members are sorted primarily based on their rank and secondly based on crowding distance. Equal to the number of people in the main population P_{t+1} members are selected from the top of the sorted list, and the rest of the members are discarded. Selected members from the next generation population, and the cycle in this section is often called the Pareto Front. None of the answers in the Pareto front are superior to each other and, depending on the circumstances, all of them can be considered as an optimal decision.

4.2 Multi-objective particles swarm optimization algorithm

Moore and Chapman (1999) developed the optimization of particle swarm for multi-objective problems [13]. Coello, et. al., in 2009 proposed an algorithm based on the

idea of an external archive [6]. Also, to select the leader, the target space is tabled. This method is described in detail in this article. To solve multi-objective problems by particle swarm optimization algorithm, it is evident that the general scheme of this algorithm needs to be modified. The primary goals while solving a multi-objective problem to achieve the maximization of the number of elements of the Pareto optimal set found, minimization of the distance of the Pareto front produced by algorithm along with maximization of the spread of solutions found. The general process of the MOPSO algorithm is described in the following steps [9]:
Step one: Create a Primary Population
Step Two: Separate nondominated members of the population and save them in archives or foreign reservoirs
Step Three: Tabling target discovered space
Step Four: Each particle chooses one leader from the archives.
Step Five: Update the velocity and position of the particles.
Each particle contains information that includes the best value so far (personal best) and the position of X^t. This information is the result of comparing the efforts that each particle makes to find the best answer. Each particle also finds the best answer so far received in the whole group, comparing the optimal values of various particles (global best). Each particle tries to change its position using the following information to achieve the best answer: 1. X^t current position, 2. V^t current velocity, 3. distance between the current and optimal personal position, and 4. distance between the current position and Pervasive optimum. Thus, the velocity of each particle and, consequently, its new position are expressed in terms of Equations (62) and (63).

$$V_i^{t+1} = wV_i^t + c_1 rand(pbest_i - X_i^t) + c_2 rand(gbest_i - X_i^t), \tag{62}$$

$$X_i^{t+1} = X_i^t + V_i^{t+1}. \tag{63}$$

In the above equation, V_i^{t+1} is the velocity of the particle i in the new repetition t, V_i^t is the velocity of the particle i in the current repetition t, X_i^t is the current position of the particle $t+1$, X_i^{t+1} is the position of the particle in the new repetition. $pbest_i$ is the best position that particle i has ever had, and $gbest_i$ is the best position of the best particle (the best position that all the particles have ever taken). $Rand$ is a random number between zero and one that is used to preserve the variety and diversity of the group. c_1 and c_2 are cognitive and social parameters, respectively. Choosing the appropriate value for these parameters will accelerate the convergence of the algorithm and prevent early convergence in local optimizations. Recent research suggests that choosing a larger value for a cognitive parameter c_1 is more appropriate than the social parameter c_2 (Khan, S. et al., 2018) [11]. The parameter w is the weighted inertia used to ensure convergence in the particle group. Weight inertia is used to control the effect of previous velocity records on current velocities.
Step 6: Use the mutation operator
Step 7: The best personal memory of each particle is updated.
Step 8: Add new nondominated members to the archive and delete the dominated members.
Step 9: Update the tabulation.

Step 10: If the stop condition is met, the algorithm stops, and the best particle among the crowd is the answer given to the problem. Otherwise, go to step four.

4.3 Multi-objective meta-heuristics algorithms comparison indices

Indicators are presented below to compare which algorithm is more applicable than another. Suppose the set of effective responses is as follows:

Most Expansion Index (MSI)

This criterion measures the expansion of the space for efficient responses. The more efficient the answers are in a wider space, the larger is the index, so the higher values of this index are intended. Suppose
f_j^{max} : The maximum value of the objective function for the purpose j among efficient responses.
f_j^{min} : The minimum value of the objective function for the purpose j among efficient responses.
This index is indicated as D and is calculated using Equation (64).

$$D = \sqrt{\sum_{j=1}^{k} (f_j^{max} - f_j^{min})^2}. \tag{64}$$

The Number of Effective responses or Pareto index (NPF)
This index indicates the number of effective responses that can be extracted using the model. Obviously, higher values for this index are preferred.

Model runtime (CPU-time)
This index shows the runtime of the model to achieve efficient responses. Obviously, the lower values for this index are preferred.

Metric Distance Index (SM)
By using this index, we will measure the uniformity of non-dominated solutions. The lower values for this index are preferred.
is calculated from Equations (65) and (66):

$$d_i = \min_{\substack{j=1,\dots,n \\ j \neq i}} \left(\sum_{k=1}^{3} |f_k^i - f_k^j| \right), \qquad i = 1, \dots, n, \tag{65}$$

$$SM = \frac{\sum_{i=1}^{n-1} |\overline{d} - d_i|}{(n-1)\overline{d}}. \tag{66}$$

5 Computational Results

5.1 Solving small sample problems by ϵ-constraint method

In this section, first, to examine the model as well as its verification, a small sample size problem is considered in accordance with the size given in Table 1. The certain and uncertain parameters considered are also generated using a uniform distribution function, based on Tables 2 and 3, to solve the problem.

Table 1: Small sample problem.

Set	Size	Set	Size
l	6	K_1	4
L	4	K_2	4
M	5	K	8
P	4	R	3
H	3	E_1	2
G	4	E_2	2
T	2	E	4

Table 2: The certain parameters are generated using a uniform distribution function.

Parameter	Interval	Parameter	Interval
$Trih_{i,h,k_2}$	$U \sim (5,10)$	$CapL_{l,t}$	$U \sim (400,450)$
$Trim_{i,m,k_2}$	$U \sim (5,10)$	$CapM_{m,t}$	$U \sim (400,450)$
$Trmh_{m,h,k_2}$	$U \sim (5,10)$	$CapH_{h,t}$	$U \sim (400,450)$
$Tril_{i,l,k_2}$	$U \sim (5,10)$	$CapG_{g,e,t}$	$U \sim (100,120)$
$Triml_{m,l,k_2}$	$U \sim (5,10)$	$CapP_{p,e,t}$	$U \sim (100,120)$
$Trgp_{g,p,k_1}$	$U \sim (5,10)$	$FixL_l$	$U \sim (50000,100000)$
$Trpi_{p,i,k_1}$	$U \sim (5,10)$	$FixM_m$	$U \sim (50000,100000)$
$Trpl_{p,l,k_1}$	$U \sim (5,10)$	$FixG_g$	$U \sim (50000,100000)$
$Hd_{e_1,t}$	$U \sim (1,3)$	$F-xP_p$	$U \sim (50000,100000)$
$NT_{k,t}$	$U \sim (10,15)$	$FixK_k$	$U \sim (100,200)$
F_{m,R_b}	$U \sim (4,5)/10$	$BigM$	10000
$Mtime1$	20	$Mtime2$	20

First and foremost, before solving the small sample problem with the ϵ-constraint method, the best and worst amount of each target function is calculated using the single optimization method. In this method, each objective function is solved, regardless of other objective functions, by using the GAMS software to determine the upper and lower boundary of each target. Therefore, it can be concluded that the generated effective responses should be between the upper and lower boundary of each target function (the best and worst values of each objective function). Table 4, named the $Payoff$ table, presents the best and worst value of each objective.

Table 3: The certain parameters are generated using a uniform distribution function.

Parameter	Pessimistic value	Most likely	Optimistic value
$\widetilde{D1}_{i,r,t}$	$U*10\sim(20,30)$	$U*10\sim(10,20)$	$U*10\sim(5,10)$
$\widetilde{D2}_{i,e_1,t}$	$U\sim(70,80)$	$U\sim(60,70)$	$U\sim(50,60)$
$\widetilde{D3}_{l,e_2,t}$	$U\sim(80,90)$	$U\sim(70,80)$	$U\sim(80,90)$
$\widetilde{T\imath h}_{i,h,k_2}$	$U\sim(25,30)$	$U\sim(15,25)$	$U\sim(10,15)$
$\widetilde{T\imath m}_{i,m,k_2}$	$U\sim(25,30)$	$U\sim(15,25)$	$U\sim(10,15)$
$\widetilde{T\imath mh}_{m,h,k_2}$	$U\sim(25,30)$	$U\sim(15,25)$	$U\sim(10,15)$
$\widetilde{T\imath gp}_{g,p,k_1}$	$U\sim(25,30)$	$U\sim(15,25)$	$U\sim(10,15)$
$\widetilde{T\imath p\imath}_{p,i,k_1}$	$U\sim(25,30)$	$U\sim(15,25)$	$U\sim(10,15)$
$\widetilde{T\imath pl}_{p,l,k_1}$	$U\sim(25,30)$	$U\sim(15,25)$	$U\sim(10,15)$
$\widetilde{Cap}_k$	$U\sim(50,55)$	$U\sim(45,50)$	$U\sim(40,45)$

Table 4: Payoff table associated with solving a small sample problem with an ϵ-constraint method.

Payoff	Z_1	Z_2	Z_3
Z_1	554028.56	4081.25	892
Z_2	1240001.14	2276.45	892
Z_3	1225043.01	4030.50	97

According to the $Payoff$ table, the best value of the first objective function is 544028.56, the best value of the second objective function is 2276.45 and the best value of the third objective function 97. Therefore, the result can be that a set of effective responses cannot provide better answers than the above. Thus, Table 5 shows the set of efficient answers from solving the small-sample sample problem by ϵ-constraint approach in the alpha-fit of 0.5.

Table 5: Set of efficient answers from solving the small-sample sample problem by ϵ-constraint method.

Efficient answer	Z_1	Z_2	Z_3
1	552568.52	2924.08	122
2	552491.45	2948.41	121
3	552401.41	2983.25	120
4	552371.14	3021.41	118
5	551604.61	3068.16	117
6	551156.80	3114.91	116
7	550809.50	3161.66	115
8	549839.27	3301.91	112
9	549093.07	3492.12	108
10	547738.31	3629.16	105
11	547293.80	3722.66	103
12	546478.20	3862.91	100
13	545933.28	4003.16	97

According to Table 5, 13 various effective answers are derived from the ϵ-constraint method for the small-sample problem

5.2 Solving small-size sample problems by meta-heuristic algorithms

In this section, due to the complexity of the problem and its NP-hardness, NSGA-II and MOPSO Meta-Heuristic Algorithms have been used to solve larger size sample problems. Therefore, in order to determine the coding accuracy, the small size sample problem presented in Table 1 has been considered and the problem has been solved by mentioned algorithms to determine the difference between the objectives functions of the meta- Heuristic algorithms and the ϵ-constraint method. Before solving the problem and analyzing the results, the initial parameters of the NSGA-II and MOPSO algorithms were adjusted by the Taguchi method. Tables 6 and 7 show the proposed levels of the parameters of these algorithms and the optimal value of each parameter obtained by theTaguchi method.

Table 6: Value of adjusted parameters (optimized) for NSGA-II.

Algorithm	Parameter	Level1	Level2	Level3	Optimized
NSGA-II	Maximum number of repetitions	50	100	200	200
	Population	50	100	200	100
	Combination rate	0.3	0.5	0.7	0.3
	Mutation rate	0.3	0.5	0.7	0.7

Table 7: Value of adjusted parameters (optimized) for the MOPSO.

Algorithm	Parameter	Level 1	Level 2	Level 3	Optimized
MOPSO	Maximum number of repetitions	50	100	200	200
	Particles	50	100	200	50
	Initial velocity coefficient	1	1.5	2	2
	Secondary velocity coefficient	1	1.5	2	1.5
	Gravity coefficient	0.8	0.9	1	0.9

After adjusting the parameters of the meta-heuristics algorithms, the small-size sample presented in Table 1 is again solved by the proposed algorithms.

After solving the small sample problem with the NSGA-II and the MOPSO algorithms, 23 effective responses for the NSGA-II algorithm and 17 effective responses for the MOPSO algorithm were obtained. According to these tables, with the increase in the number of relief and cargo vehicles, the total cost of the supply chain network design is increased and, consequently, due to the transfer of more injured to medical centers, etc., the amount of shortage or lack of service to the injured people is reduced. After solving the small size problem with different methods, for the purpose of evaluating the effective response indicators, Table 8 is created. In this table, the averages of the target functions, the number of efficient responses, the most expansion index, the metric distance index, and the computation time are considered.

Table 8: Comparison of the indicators of the affective responses is the solving of the small size problem.

	Solving methods		
Indexes	MOPSO	NSGA-II	GAMS(ϵ-constraint)
Mean of Z_1	621766.78	624332.30	549983.02
Mean of Z_2	2732.39	2786.55	3325.67
Mean of Z_3	127.82	123.82	111.84
NPF	17	23	13
MSI	111452.59	188440.97	6277.45
SM	0.79	0.78	0.546
CPU-Time	67.31	58.39	267.16

According to Table 8, it can be concluded that if one specific problem-solving method has the lowest values in the index of target functions averages, the metric distance, and the computation time, and has the higher values in the number of effective responses index and the distance index, the most efficient solution method is obtained. By examining the results of Table 8, the ϵ-constraint method is better than other methods in the indexes of the average of the first and third objective functions and the metric distance index. The NSGA-II algorithm has been better in acquiring the number of efficient responses, the most extension, and computation time indexes, and the MOPSO algorithm has also been better considering the mean index of the second-objective function.

5.3 Solving larger size problems

In this section, due to the inadequacy of the GAMS software and the ϵ-constraint method to solving relief logistics problems, only the NSGA-II and MOPSO algorithms are used to solve the problem in larger sizes. In this section, the large sample problems have been reviewed at three levels: small, medium, and large. Therefore, 15 sample problems are designed based on Table 9 and generated data according to Tables 2 and 3.

Table 9: The large sample size problems.

Set	Sample Problems														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
L	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
M	10	10	12	12	14	14	16	16	18	18	20	20	22	25	25
P	6	6	6	6	8	8	8	8	10	10	10	10	12	12	12
H	4	4	4	4	4	6	6	6	6	8	8	8	8	12	12
G	6	6	6	6	8	8	8	8	10	10	10	12	12	12	12
T	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6
K_1	6	6	6	6	8	8	8	8	10	10	10	12	12	12	12
K_2	6	6	6	6	8	8	8	8	10	10	10	12	12	12	12
K	12	12	12	12	16	16	16	16	20	20	20	24	24	24	24
R	4	4	4	4	5	5	5	6	6	6	6	7	7	7	8
E_1	3	3	3	3	3	4	4	4	4	5	5	5	6	6	6
E_2	3	3	3	3	3	4	4	4	4	5	5	5	6	6	6
E	6	6	6	6	6	8	8	8	8	10	10	10	12	12	12

In this section, sample problems from 1 to 15 are reviewed. As stated, the data used to solve these problems is taken from Tables 2 and 3. In addition to this information, in all of the problems examined, the alpha fitting level is assumed to be 0.5. Table 10 shows the average of the effective responses and comparison indicators of the meta-heuristics algorithms for the solving of larger sample size problems.

T-test was used at 95% confidence level to examine the significant difference between the obtained averages in different indices in solving larger sample size problems. Therefore, considering the confidence level if the P test statistic is less than 0.05, there is a significant difference between the mean of that computational index. On the other hand, if the value of P test is more than 0.05, there is no significant difference between the computing index averages. Table 11 summarizes the results of the T-test test among the averages of the indices used in Table 11 for larger sample size problems.

Regarding the value of the P test obtained from Table 11, it can be concluded that there is no significant difference between any of the averages of the studied indices in larger sample size problems. Therefore, the multi-objective multi-factor TOPSIS decision-making method has been used to conclude on the most efficient algorithm in solving sample size larger. Table 12 summarizes the results of the indexes obtained from solving sample problems by the larger size and the amount of utility weight. Obviously, the greater the weight of the utility, the higher the efficiency of the algorithm in solving the larger sample problem, considering all the indices.

According to Table 12, the NSGA-II algorithm is more efficient than the MOPSO algorithm with a larger utility weight (0.8054) in solving larger size sample problems considering all comparison criteria.

Table 10: Comparison of multi-objective multi-heuristic algorithms indicators in solving larger-size problems.

Method	Problem	Mean of Z_1	Mean of Z_2	Mean of Z_3	SM	MSI	NPF	CPU Time
NSGA-II	1	814624.5	4034.91	212.0	0.37	270273.91	19	66.64
	2	893903.9	4037.42	250.2	0.77	585593.25	19	109.5
	3	1054091.5	4499.47	305.6	0.7	479316.63	20	173.8
	4	1340197.5	6188.69	554.4	0.57	850298.87	14	244.0
	5	1499104.9	7570.90	613.0	0.41	1290789.7	14	332.4
	6	1819457.7	10908.06	1115.2	0.67	2508017.5	26	432.4
	7	11904795.9	13175.46	1199.2	0.52	2797218.3	24	543.0
	8	2147039.9	14072.13	1318.8	0.48	2526486.4	19	661.9
	9	2275232.2	14799.79	1376.8	0.85	2489246.8	19	814.0
	10	2821578.6	20513.16	2191.6	0.39	350950.3	23	956.4
	11	2977894.9	21975.99	2315.00	0.76	3087180.76	26	1039
	12	3196969.1	24808.73	2470.40	0.45	4883033.12	27	1324
	13	3869518.3	33011.01	3557.80	0.97	3839428	32	1530
	14	4233689.6	34484.42	3725.60	1.03	4565022.62	29	1807
	15	4451831.3	36058.92	3939.00	0.68	5381370.91	27	2640
MOPSO	1	821831.2	3944.16	207.8	0.46	109850.13	18	75.39
	2	922036.1	4152.08	246.4	0.62	329845.53	14	90.49
	3	1055897.5	4493.01	296.8	0.23	370471.43	18	111.8
	4	1334944.5	6423.04	543.0	0.59	463108.57	16	133.3
	5	1513069.6	7370.91	625.4	0.35	817523.73	18	261.6
	6	1837999.3	11194.19	1096.6	0.55	2008648.7	31	345.4
	7	1922322.9	12984.41	1181.0	0.59	2559860.1	28	494.9
	8	2109015.3	13969.09	1294.4	0.84	3694417.3	19	723.4
	9	2259144.9	14816.02	1375.8	0.69	2215210.1	15	982.0
	10	2835361.1	20534.23	2167.4	0.36	2437807.6	25	1326
	11	2979000.7	21726.00	2308.00	0.94	2437807.91	32	1325
	12	3204856.4	24920.01	2463.80	0.86	3887334.58	19	1834
	13	3865944.8	3293.58	3556.20	0.36	3757576.16	24	2340
	14	4244230.4	34458.61	3660.40	0.47	4595983.26	21	2978
	15	4459728.0	36441.11	3925.60	1.03	5928298.63	20	3953

Table 11: Statistical T-test results at 95% confidence level for the indexes in larger-size problems.

Index	Mean difference	Lower bound	Upper bound	P	T
Z_1	4364	−891413	900140	0.992	0.01
Z_2	15	−8455	8485	0.997	0.01
Z_3	13	−952	978	0.978	0.03
NPF	1.33	−2.77	5.44	0.511	0.67
MSI	230699	−1046492	1507890	0.714	0.37
SM	0.045	−0.1222	0.2128	0.583	0.56
CPU-Time	287	−455	1029	0.432	0.80

Table 12: The most effective meta-heuristic algorithm for solving sample problems of larger size.

Algorithm	$Z1$	$Z2$	$Z3$	SM	MSI	NPF	CPU-Time	Utility weight
NSGA-II	2353329	16675.94	1676.30	0.641	2604948	22.53	844.816	0.8054
MOPSO	2357629	16690.83	1663.24	0.596	2374250	21.20	1131.61	0.1946
Index weight	0.2	0.2	0.2	0.1	0.1	0.1	0.1	

6 Conclusions and Suggestions for Future Studies

This paper presents a multi-objective model for the multi-period location-routing problem, taking into account the evacuation of casualties and homeless people and fuzzy paths in relief logistics. First, an uncertain multi-objective model of the problem was designed under uncertain parameters of demand, time, and transport capacity, and then, using the fuzzy programming method, uncertain parameters of the problem were controlled. Considering the multi-purpose design of the model, a small sample size was first designed, and the model was solved using the ϵ-constraint method in GAMS software, resulting in 13 different efficient responses. Then, due to the NP-hardness of the problem and the inability of GAMS software to solve the model in larger sizes NSGA-II and MOPSO meta-heuristic algorithms were used to solve the problem. At first, the small sample size problem solved by the GAMS software was solved by these algorithms, which showed the high efficiency of the algorithms in obtaining efficient responses. Then, 15 sample problems were designed in larger sizes, and sample problems were analyzed in 5 successive replications by the NSGA-II and MOPSO algorithms. Before solving sample problems in larger sizes, the initial parameters of both algorithms were adjusted by the Taguchi method so that the algorithms have the highest efficiency in obtaining results. The results showed that there was no significant difference between all indices of the case. The indexes computed in this problem include the mean of target functions, the number of efficient responses, the most exponential index, the metric distance index, and computational time. Due to the lack of decision about choosing the most efficient algorithm, the TOPSIS multi-criteria decision-making method was used, which resulted in the selection of the NSGA-II algorithm with a utility weight of

0.8054 compared to the MOPSO algorithm with a utility weight of 0.1946 in solving all sample problems. In this regard, the followings are suggestions for other researchers:

1. Considering the transportation cost parameter to be uncertain.
2. Using robust fuzzy optimization method to control uncertain parameters due to lack of access to historical data
3. Use of other meta-heuristic algorithms such as MOSA, MOALO to solve problems
4. We were considering the reliable objective function along with the mentioned target functions in this study.

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Numerical Solution of Vasicek Equation by Using Brownian Wavelets and Multiple Ito-Integral

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Abstract. In this paper, we present a new approach to solving stochastic differential equations and the Vasicek equation by using Brownian wavelets and multiple Ito-integral. Firstly, the calculation of the multiple Ito-integral based on the structure of Brownian motion is presented and the error of Ito-integrate computation is minimized under this condition. Then, the Brownian wavelets 1D and 3D based on coefficients Brownian motion are introduced. After that, a system of linear and nonlinear equations of coefficients Brownian motion is obtained such that by solving this system the approximate solution of the Vasicek equation is obtained. In the last section, some numerical examples are given.

Keywords. Stochastic differential equation, Vasicek equation, Brownian motion, Brownian wavelets, Ito-integral.

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1 Introduction

For the First time, microscopic Brownian motion (abbreviated to BM) was discovered by Robert Brown a scientist who observed through a microscope the random swarming motion of pollen grains in water. The theory of Brownian motion was developed by Bachelier [3], also by Einstein in his paper [4], which used Brownian motion to estimate Avogadro's number. Now, BM also called the Wiener process has been defined by Wiener in 1923 [14]. Wiener proved that there exists a special case of Brownian motion with continuous paths in general form; Brownian motion is a Gaussian process and a Markov process. So, it is in the theory of stochastic process. Also history of Brownian motion and related processes we cite Meyer, Kahane [6], [7], and Yor [11]. Firstly, Vasicek was accounting for a bond pricing model that interest rate. The short rate dynamics is defined as a diffusion process with constant parameters [10]. The bond price is based on some assumptions; it has the feature that on a given date, the market price of risk. Vasicek's model is a special case of the Ornstein-Uhlenbeck (O-U) process, with constant volatility. This implies that the short rate is both a Gaussian process and a Markovian process. Vasicek's model also exhibits mean-reversion and is able to capture monetary authority's behavior of setting target rates. Also, the historical experience of interest rates justifies the Ornstein-Uhlenbeck specification. So, the pedagogical value of this model in stochastic term structure modeling is considerable. Also, there are many problems that can be modeled as stochastic differential equation (SDE). In this study, we present a new numerical method for solving stochastic differential equations. Firstly, we introduce some definition and numerical approximation of Ito-integral based on Brownian motion after that, the Brownian wavelets 1D and 3D-Brownian motion based on coefficients Brownian motion are introduced. By using this result, a system of linear and nonlinear equations of coefficients Brownian motion is obtained such that by solving this system the approximation solution of the Vasicek equation is optioned.

2 Basic Definitions and Theorems

The Karhunen-Loeve expansion operates much the same as Fourier Expansion, but on the space $L^2(\omega, F, P)$ of equivalence classes of random variables is a Hilbert space, and the inner product $E[X_1 X_2]$, also inducted norm $X_2 = \sqrt{E(X^2)}$ and random functions X_t that is continuous-parameter, $t \in [a, b]$ considered over $[a, b]$. These random functions are found by study the Hilbert-Schmidt integral operator, which has the covariance of the random function as a kernel. This expansion decomposes the stochastic process by projecting every variable onto an orthonormal basis for space spanned by the operator's Eigen-functions, which equivalence of the positive eigenvalues thereof. The coefficients of the infinite linear combination are expected to be random variables, hence, for all $t \in [a, b]$, they are the projection of a random variable onto a deterministic orthogonal basis. As a result, these random coefficients are also orthogonal in $L^2(\omega, F, P)$, namely, they are uncorrelated.

Theorem 1 (The Karhunen-Loeve Expansion)**.** Let X_t be a zero-mean square-integrable stochastic process defined over a probability space (ω, F, P) and indexed over a closed and bounded interval $[a, b]$, with continuous covariance function $k(s, t)$. Then $k(s, t)$ is a Mercer kernel and letting e_k be an orthonormal basis on $L^2([a, b])$ formed by the eigen functions of T_{kx} with respective eigenvalues λ_k, X_t admits the following representation

$$x_t = \sum_{i=1}^{\infty} z_i e_i(t) \quad in \quad L^2(\omega),$$

where the convergence is in L^2 uniform in t and $Z_i(\omega) = \int_a^b x_t(\omega) e_i(t) dt$.

Furthermore, the random variables Z_k have zero-mean, are uncorrelated and have variance λ_k, i.e., we have:

- $E(z_i) = 0$.
- $E(z_i z_j) = \delta_{ij} \lambda_j$ where δ_{ij} is Kronecker delta.
- $Var(z_i) = \lambda_j$.

Note that by generalizations of Mercer's theorem we can replace the interval $[a, b]$ with other compact spaces C and the Lebesgue measure on $[a, b]$ with a Borel measure whose support is C.

Proof. See [5]. □

3 Brownian Wavelet

In this section, we introduce the Brownian wavelet and some another definition. In the first (see [12]), we fix the time interval $[t_0, t_1]$. Therefore, the standard Brownian motion has a Karhunen–Loeve series expansion of the form such that the $z_i \sim N(0, 1)$, $i = 1, 2, \cdots$ are independent random variables and ψ_i are orthonormal bases of the Hilbert space with inner product as:

$$< f, g >= \int_{t_0}^{t_1} f(\tau) g(\tau) d\tau.$$

The Gaussian random variables are the projections of the Brownian motion onto the basic functions

$$z_i = \int_{t_0}^{t} \psi_i(\tau) d\beta(\tau).$$

The series expansion can be interpreted as the following representation for the differential of Brownian motion

$$d\beta(t) = \sum_{i=0}^{\infty} z_i \psi_t dt.$$

We have the following theorem.

Theorem 2. Stochastic processes $\beta(t)$ that has series expansion of this form, is a Brownian motion.

$$\beta(t,\omega) = \sum_{n=0}^{\infty} a_n \psi_n(t) G_n(\omega). \tag{1}$$

Proof. See [12]. □

As we know, the basic functions are introduced in various forms and here we use the following form:

$$\beta(t) = \sum_{i=0}^{\infty} z_i \int_{t_0}^{t} \psi_i(\tau) d\beta(\tau),$$

where the functions $\psi_i(t),\ i = 1, 2, 3, \cdots$, are introduced in the following:

Definition 1. (Piecewise function): We define the $\psi_i(t)$ as follow:

$$\psi_1(t) = \begin{cases} 2t & 0 \le t < 1/2, \\ 2 - 2t & 1/2 \le t < 1, \\ 0 & \text{o.w.} \end{cases}$$

$$\psi_n = \psi_1(2^j t - k),$$

where $t \in [0,1]$ and for $n > 1,\ n = 2^j + k$. Some of $\psi_n(t)$ heve been shown in Figures 1 and 2.

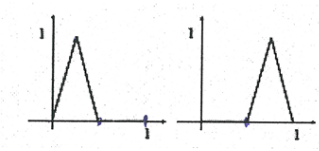

Figure 1: $\psi_2(t), \psi_3(t)$.

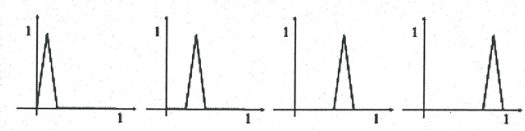

Figure 2: $\psi_4(t), \psi_5(t), \psi_6(t), \psi_7(t)$.

Definition 2. (Sinus function): We consider a well-defined

$$\psi_n(t) = \sqrt{2}\sin[(n - 1/2)\pi t]. \tag{2}$$

Some other $\psi_n(t)$ heve been shown in Figures 3 and 4, where $1 \le n$.

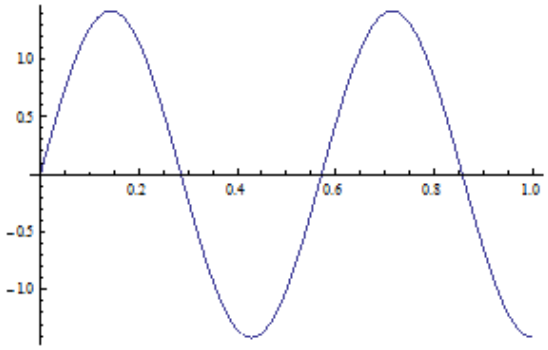

Figure 3: $\psi_3(t) = \sqrt{2}\sin[(5/2)\pi t]$.

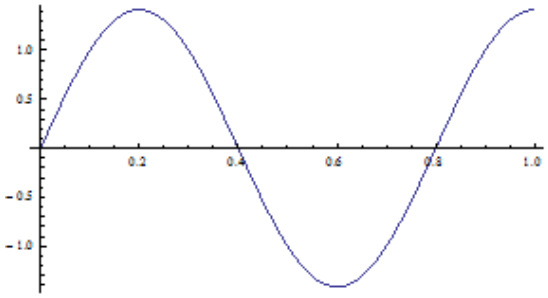

Figure 4: $\psi_4(t) = \sqrt{2}\sin[(7/2)\pi t]$.

3.1 1D and 3D- Brownian Wavelet

The one-dimensional Brownian wavelets can be constructed as basic sinus functions $\psi_n(t)$, such that are orthonormal, so we have the following equations:

$$< \psi_n, \psi_{n-1} >= 0,$$

$$< \psi_n, \psi_n >= 1.$$

In [12], it has been represented that the wavelet coefficients are

$$a_n = \frac{1}{(n-1/2)\pi} \qquad n = 1, 2, \cdots, . \tag{3}$$

The three-dimensional wavelets [10] can be constructed as separable products of 1D-wavelets by applying 1D-wavelets in three dimensional and three directions (x, y, z). The volume $F(x, y, z)$ at first is filtered the x-direction, resulting in a high-pass image $H(x, y, z)$ and a low-pass image $L(x, y, z)$. Both of the low-pass and high-pass are then filtered along the y-direction, resulting in four decomposed sub-volumes: Low-Low, Low-High, High-Low and High-High. Then each of these four sub-volumes are filtered along the z-direction, resulting in eight sub volumes: Low-Low-Low, Low-Low-High, Low-High-Low, Low-High-High, High-Low-Low, High-Low-High, High-High-Low and High-High-High. Similar to one dimensional case, 3D-Brownian wavelet can be constructed by 3D basic functions, $\psi_n(s, t)$. these functions are defined by

$$\psi_n(s,t) = 2\sin[(n-1/2)\pi s]\sin[(n-1/2)\pi t]. \tag{4}$$

As we know, these functions are orthonormal, so we have:

$$< \psi_n(s,t), \psi_{n-1}(s,t) >= 0,$$
$$< \psi_n(s,t), \psi_n(s,t) >= 1.$$

Some generated functions are shown in the Figures 5 and 6.

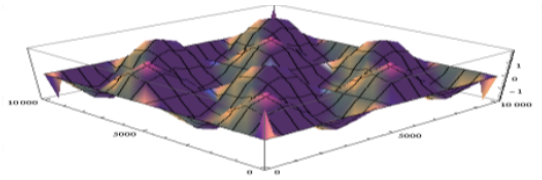

Figure 5: $\psi_3(s,t) = 2\sin[(5/2)\pi s]\sin[(5/2)\pi t]$.

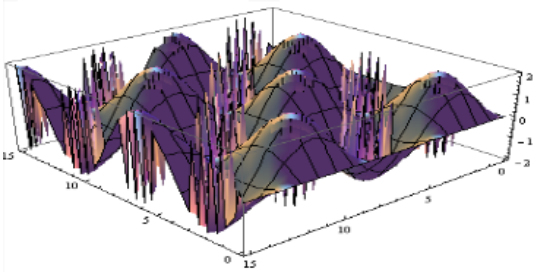

Figure 6: $\psi_4(s,t) = 2\sin[(7/2)\pi s]\sin[(7/2)\pi t]$.

The wavelet coefficients are as follow:

$$\bar{a_n} = 2\sqrt{2}a_n^2, \qquad a_n = \frac{1}{(n-1/2)\pi}. \tag{5}$$

4 Numerical Approximation of Multiple Ito-integral

There are two steps in defining a double Ito-integral. The first step is to define the integral for "off-diagonal step functions" and the second step is to approximation a function in $L^2([a,b]^2)$ by "off-diagonal step functions" then, take the limit of corresponding integrals. To motivate the notion of an "off-diagonal step function" and its necessity, consider the example of defining the double Ito-integral $\iint 1 d\beta(t)\beta d(s)$ Let $\Delta_n = \{t_0, t_1, \cdots, t_n\}$ be a partition of the interval $[0,1]$. Let, the partition of the interval square $[0,1)^2$ be as follow:

$$[0,1)^2 = \bigcup_{i,j=1}^{n}[t_{i-1}, t_i) \times [t_{j-1}, t_j).$$

Then we get the following Riemann-sum for the integral $f \equiv 1$:

$$\sum_{i,j=1}^{n}(\beta_{t_i} - \beta_{t_{i-1}})(\beta_{t_j} - \beta_{t_{j-1}}) = \sum_{i=1}^{n}(\beta_{t_i} - \beta_{t_{i-1}})^2 = \beta(1)^2.$$

4.1 Numerical approximation of Ito-integral

Now, let us consider the following integral:

$$\int_a^b f(t)d\beta(t) = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})(\beta_{t_i} - \beta_{t_{i-1}}),$$

where $f(t)$ is a Brownian motion

$$\int_a^b \beta(t)d\beta(t) = \sum_{n=1}^{\infty} \beta_{n-1}(\beta_n - \beta_{n-1}).$$

In [10] we know that $\beta(t,\omega) = \sum_{n=1}^{\infty} a_n\psi_n(t)G_n(\omega)$, where $G_n(\omega) \in [0,1]$ and $\psi_n(t), a_n$ have been defined in [14]. Without losing the generality, for $G_n(\omega)$ has the same value for both of $\beta(t,\omega)$,

$$\int_a^b \beta(t)d\beta(t) = G_n(\omega)\{\sum_{n=1}^{\infty} a_n\psi_n(t_{n-1})[\sum_{n=1}^{\infty} a_n\psi_n(t_n) - \sum_{n=1}^{\infty} a_n\psi_n(t_{n-1})]\}.$$

For a given value of $G_n(\omega)$ the following approximation can be established

$$\int_a^b \beta(t)d\beta(t) = G_n(\omega)\{-\sum_{n=1}^{\infty} a_n^2\psi_n(t_{n-1})^2\}. \tag{6}$$

Now, some examples are illustrated.

4.2 Numerical examples

By considering the above discussion two examples are given:

Example 1. We calculate the Ito-integral $\int_0^1 \beta^2(t)d\beta(t)$. We know

$$\int_a^b \beta(t)d\beta(t) = G_n(\omega)\{-\sum_{n=1}^{\infty} a_n^2\psi_n^2(t_{n-1})\}.$$

So for

$$\int_0^1 \beta^2(t)d\beta(t) = G_n(\omega)\{-\sum_{n=1}^{\infty} a_n^3\psi_n^3(t_{n-1})\},$$

for $G_n(\omega) = 0.1$,

$$0.1\{-\sum_{n=1}^{\infty} a_n^2\psi_n^2(t_{n-1})[\sum_{n=1}^{\infty} a_n\sqrt{2}\sin[(n-1/2)\pi t] - \sum_{n=1}^{\infty} a_n\sqrt{2}\sin[((n-1)1/2)\pi t]\}.$$

Now for different random value $G_n(\omega)$, numerical solutions of Ito-integral are shown in Table 1.

Table 1: Numerical solutions of Ito-integral for different random value $G_n(\omega)$.

$G_n(\omega)$	**Ito-integral approximation**
0.1	0.00253019
0.2	0.00506037
0.3	0.00759056
0.4	0.0101207
0.5	0.0126509
0.6	0.0151811
0.7	0.0177113
0.8	0.0202415
0.9	0.0227717
1	0.0253019

Now, consider the following integral:

$$\iint_a^b \beta(t,s,\omega)d\beta(t,\omega)d\beta(s,\omega),$$

where $\beta(t,s,\omega)$ is a 3D-Brownian motion. Notice to $\tilde{G}_n(\omega) \in [0,1]^2$ we have

$$\int_a^b \beta(t,\omega)[\int_a^b \beta(s,\omega)d\beta(s,\omega)]d\beta(t,\omega).$$

Now, some examples are given.

Example 2. We consider the Ito-integral $\iint_0^1 \beta^2(t,s,\omega)d\beta(t,\omega)d\beta(s,\omega)$. Similar to pervious algorithm for two steps and for different random value $\tilde{G}_n(\omega)$, we mentioned the numerical solutions in Table 2.

Table 2: Numerical solutions of multiple Ito-integral for different value $\tilde{G}_n(\omega)$.

$\tilde{G}_n$	**Multiple Ito-integral approximation**
0.01	6.40186×10^{-6}
0.02	0.0000256073
0.03	0.0000576166
0.04	0.000102429
0.05	0.000160045
0.06	0.000230466
0.07	0.00031369
0.08	0.000409718
0.09	0.00051855
1	0.000640186

5 Numerical Solution of Vasicek Equation

Firstly, consider a linear stochastic differential equation in the general form as follow:

$$\frac{dx}{dt} = f(x,t) + L(x,t)\omega(x,t),$$

where the initial conditions are $x(t_0) \sim N(m_0, P_0)$ and the $f(t)$ and $L(t)$ are matrix valued functions of time, and $w(t)$ is white noise. New research contributes to the development of the mathematical ways in solving of the bond price and interest rate under the Vasicek model. In modeling the indeterminable of interest rates, consider that there is a probability space (ω, F, P) with a standard filtration F_t. Under the risk-neutral measure P the short rate dynamics is defined as:

$$dx(t) = -x(t)dt + \sigma d\beta(t), \tag{7}$$

where σ is a positive constant. We have

$$\dot{x}(t) = -x(t) + \sigma\omega(t). \tag{8}$$

Now let we assume $x(t)$ is Brownian motion, and consider $a_n G_n = \tilde{a}_n$,

$$x(t) = \sum_{n=0}^{\infty} \tilde{a_n}\psi_n(t). \tag{9}$$

We get

$$\dot{x(t)} = \sum_{n=0}^{\infty} \tilde{a_n}\dot{\psi_n}(t). \tag{10}$$

By multiplying the relation (8) by $\psi_n(t)$ and integrating the sides, we have:

$$\int_1^2 \dot{x(t)}\psi_n(t)dt = -\int_1^2 x(t)\psi_n(t)dt + \sigma\int_1^2 \psi_n(t)\omega(t)dt. \tag{11}$$

We know that $\omega(t)d(t) = d\beta(t)$. So, we get

$$\int_1^2 x(t)(\psi_n(t) - \dot{\psi_n}(t))dt = \sigma\int_1^2 \psi_n(t)d\beta(t). \tag{12}$$

Substituting (9) and (10) into (12) gives

$$\int_1^2 \tilde{a_n}\psi_n(t)(\psi_n(t) - \dot{\psi_n}(t))dt = \sigma\int_1^2 \psi_n(t)d\beta(t).$$

For $\sigma = 1$, we know $< \psi_n, \psi_n >= 1$. So,

$$2\tilde{a_n} = \int_1^2 \psi_n(t)d\beta(t). \tag{13}$$

Now, according to numerical approximation Ito-integral method, we can obtain the following relation.

$$2\tilde{a_n} = \sum_{n=1}^{\infty} a_n [\sum_{i=1}^{\infty} \psi_n(t_{i-1})\psi_n(t_i) - i^2].$$

Without losing the generality, let $t_i = 1, t_{i-1} = i - 1 \quad (i = 1, 2, \cdots)$. Therefore, for $t_1 = 1$ and random value 0.1, we get

$$2\tilde{a_n} = \sum_{n=1}^{\infty} a_n [\psi_n(1)\psi_n(0) - 1].$$

For $n = 3$, the following system of equations is obtained:

$$\begin{aligned}
2\tilde{a_1} &= a_1[\psi_1(1)\psi_1(0) - 1], \\
2\tilde{a_2} &= a_1[\psi_1(1)\psi_1(0) - 1] + a_2[\psi_2(1)\psi_2(0) - 1], \\
2\tilde{a_3} &= a_1[\psi_1(1)\psi_1(0) - 1] + a_2[\psi_2(1)\psi_2(0) - 1] + a_3[\psi_3(1)\psi_3(0) - 1].
\end{aligned}$$

This can be summarized as the following matrix form.

$$\left[\begin{array}{c|c} \tilde{a_1} & {}^{1}\!/_{10\pi} \\ \tilde{a_2} & {}^{1}\!/_{5\pi} \\ \tilde{a_3} & {}^{1}\!/_{10\pi} \end{array}\right].$$

Table 3: Ito-integral approximation for $n = 3$, $t_1 = 1$ and different values a_n.

a_n	$\tilde{a_1}$	$\tilde{a_2}$	$\tilde{a_3}$
0.1	${}^{1}\!/_{10\pi}$	${}^{1}\!/_{5\pi}$	${}^{1}\!/_{10\pi}$
0.2	${}^{1}\!/_{5\pi}$	${}^{2}\!/_{5\pi}$	${}^{1}\!/_{5\pi}$
0.3	${}^{3}\!/_{10\pi}$	${}^{3}\!/_{5\pi}$	${}^{3}\!/_{10\pi}$
0.4	${}^{2}\!/_{5\pi}$	${}^{4}\!/_{5\pi}$	${}^{2}\!/_{5\pi}$
0.5	${}^{1}\!/_{2\pi}$	${}^{1}\!/_{\pi}$	${}^{1}\!/_{2\pi}$
0.6	${}^{3}\!/_{5\pi}$	${}^{6}\!/_{5\pi}$	${}^{3}\!/_{5\pi}$
0.7	${}^{7}\!/_{10\pi}$	${}^{7}\!/_{5\pi}$	${}^{7}\!/_{10\pi}$
0.8	${}^{4}\!/_{5\pi}$	${}^{8}\!/_{5\pi}$	${}^{4}\!/_{5\pi}$
0.9	${}^{9}\!/_{10\pi}$	${}^{9}\!/_{5\pi}$	${}^{9}\!/_{10\pi}$

Therefore, the corresponding results for another random value, i.e., $n = 3$ and $t_2 = 2$, are available. This can be summarized as the following matrix form that is given in Table 4.

Table 4: Ito-integral approximation for $n = 3$, $t_2 = 2$ and different values a_n.

a_n	$\tilde{a_1}$	$\tilde{a_2}$	$\tilde{a_3}$
0.1	$1/2\pi$	$1/2\pi$	$1/2\pi$
0.2	$1/\pi$	$2/\pi$	$1/\pi$
0.3	$3/2\pi$	$3/\pi$	$3/2\pi$
0.4	$2/\pi$	$4/\pi$	$2/\pi$
0.5	$5/2\pi$	$5/\pi$	$5/2\pi$
0.6	$3/\pi$	$6/\pi$	$3/\pi$
0.7	$7/2\pi$	$7/\pi$	$7/2\pi$
0.8	$4/\pi$	$8/\pi$	$4/\pi$
0.9	$9/2\pi$	$9/\pi$	$9/2\pi$

Finally, the numerical Ito-integral for $n = 3$ can be resulted in the following form:

$$a_1 = 0.1 \quad \rightarrow x(t_1) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t) = \frac{1}{10\pi}\psi_1(t) + \frac{1}{5\pi}\psi_2(t) + \frac{1}{10\pi}\psi_3(t).$$

$$a_2 = 0.2 \quad \rightarrow x(t_1) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t) = \frac{1}{5\pi}\psi_1(t) + \frac{2}{5\pi}\psi_2(t) + \frac{1}{5\pi}\psi_3(t).$$

Also, the other coefficients can be obtained by the equation (12). One of the advantages of this approach is that we can extend this relation and obtain the weights with constant values for $\tilde{a_n}$. Now, let we define the 3D-Vasicek equation by 3D-Brownian motion. This 3D-stochastic partial differential equation has the general form:

$$\frac{\partial^2 x(t,s)}{\partial t \partial s} = -x(t,s) + \sigma w(t,s), \tag{14}$$

where $x(t,s)$ is a 3D-Brownian motion and consider

$$x(t,s) = \sum_{n=0}^{\infty} \tilde{a_n}^2 \psi_n(t,s), \tag{15}$$

we get

$$\dot{x}(t,s) = \sum_{n=0}^{\infty} \tilde{a_n}^2 \dot{\psi}_n(t,s), \tag{16}$$

where $\psi_n(t,s)$ are basic functions of 3D-stochastic wavelet. Multiply the relation (14) by $\psi_n(t,s)$ and integrate the sides, the relation (17) is derived.

$$\iint_1^2 \dot{x}(t,s)\psi_n(t,s)dtds = -\iint_1^2 x(t,s)\psi_n(t,s)dtds + \sigma \iint_1^2 \psi_n(t,s)\omega(t,s)dtds. \tag{17}$$

We know that $\omega(t)\omega(s)dtds = d\beta(t,s)$. So, we know that

$$\iint_1^2 x(t,s)(\psi_n(t,s)-\psi_n(\dot{t},s))dtds = \sigma \iint_1^2 \psi_n(t,s)d\beta(t)d\beta(s), \tag{18}$$

and Hence, similar to (13) and substituting into (15) and also (16) into (18) and for $\sigma = 1$, the relation (19) is resulted.

$$4\tilde{a_n}^2 = \iint_1^2 \psi_n(t,s)d\beta(t)d\beta(s). \tag{19}$$

Now, according to the numerical approximation multiple Ito-integral method

$$4\tilde{a_n}^2 = \sum_{n=1}^{\infty} a_n[\sum_{i=1}^{\infty} \psi_n(t_{i-1})\psi_n(t_i) - i^2] \sum_{n=1}^{\infty} a_n[\sum_{j=1}^{\infty} \psi_n(s_{j-1})\psi_n(s_j) - j^2],$$

the coefficients matrix Vasicek is obtained:

$$\mathbf{V}_{ij} = \frac{ia_j}{\pi} \qquad j,i = 1,2,\cdots.$$

Finally, the numerical multiple Ito-integration for $n = 3$ and t_1, s_1 takes the following form:

$$a_1 = 0.1 \quad \to x(t_1,s_1) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t,s) = \frac{1}{10\pi}\psi_1(t,s) + \frac{1}{5\pi}\psi_2(t,s) + \frac{1}{10\pi}\psi_3(t,s),$$

$$a_2 = 0.2 \quad \to x(t_1,s_1) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t,s) = \frac{1}{5\pi}\psi_1(t,s) + \frac{2}{5\pi}\psi_2(t,s) + \frac{1}{5\pi}\psi_3(t,s).$$

The numerical multiple Ito-integral for $n = 3$ and t_2, s_2 takes the following form

$$a_1 = 0.1 \quad \to x(t_2,s_2) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t,s) = \frac{1}{2\pi}\psi_1(t,s) + \frac{1}{\pi}\psi_2(t,s) + \frac{1}{\pi}\psi_3(t,s),$$

$$a_2 = 0.2 \quad \to x(t_2,s_2) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t,s) = \frac{1}{\pi}\psi_1(t,s) + \frac{2}{\pi}\psi_2(t,s) + \frac{1}{\pi}\psi_3(t,s).$$

6 Numerical Results

In this section, we compare the proposed numerical solution of the 3D-Vasicek equation by using Brownian wavelets with the other existing methods for this class of problems [9], [12] path-wise variable time-stepping method, Quasi-symplectic method, Midpoint rule, and Runge-Kutta for three-dimensions. The methods are applied to some models of interest, which have a relative interest rate. In each example, we ran the proposed algorithm with a given random variable, in order to compare their efficiency. We compute the error ϵ between numerical approximation and semi exact solution of the 3D-Vasicek equation. The results are shown in Table 5.

The result of the simulation for 3D-Vasicek equation by using Brownian wavelet regarding the obtained coefficients are shown in Figures 7 and 8.

Table 5: The error between numerical approximation and semi exact solution of 3D-Vasicek.

method	$\tilde{a_n} = 0.1$	$\tilde{a_n} = 0.5$	$\tilde{a_n} = 1$
Brownian wavelet	0.0275	0.003309	0.035051
Midpoint rule	0.219	0.445	0.086
Runge-Kutta	0.187	0.784	0.046
Pathwise variable time-stepping	0.3647	1.00028	0.9968
Quasi-symplectic	0.1796	0.883	0.092

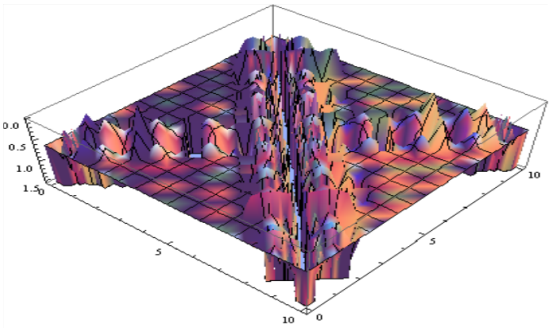

Figure 7: Simulation for 3D-Vasicek equation by using Brownian wavelet for $\tilde{a_n} = 0.2$, $n = 5$, $t_1 = 1$.

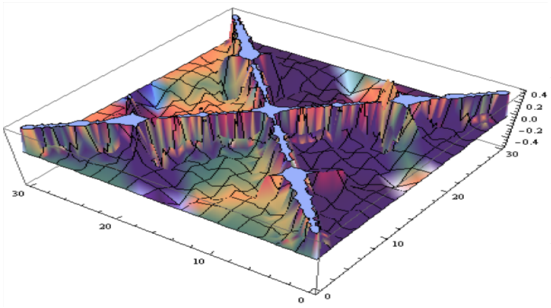

Figure 8: Simulation for 3D-Vasicek equation by using Brownian wavelet for $\tilde{a_n} = 0.5$, $n = 10$, $t_1 = 1$.

7 Conclusion

In this paper, a new approach for solving a stochastic differential equation is presented and the numerical approximate solution of the Vasicek equation is obtained. Firstly, the numerical approximation of Ito-integral based on Brownian motion is introduced. Then, the obtained algorithm evaluated the random value based on $a_n G_n = \tilde{a}_n$, and this procedure is applicable for solving multiple Ito-integral. Afterward, the 1D and 3D-Brownian wavelets based on coefficients Brownian motion are introduced. By using the obtained results, a system of linear and nonlinear equations of coefficients Brownian motion is obtained, such that by solving this system, the approximate solution of the Vasicek equation is obtained. In the end, some numerical examples are given.

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An Efficient Data Collection Algorithm to Estimate Unknown Target Parameter in Wireless Sensor Networks

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Abstract. Estimating the target parameter while the prior distribution function is known, and several observations which are provided by the sensor node is the main goal in this paper. In wireless sensor networks (WSN), nodes sense the environment and send data to a sink node called Fusion Center (FC). FC collects data and estimates the observed parameter with user-defined precision. The proposed algorithm increases network lifetime and has an efficient estimation process. For this purpose, the proposed algorithm schedules node's activity and determines the multihop path between nodes and FC. Simulation and performance analysis demonstrates proposed algorithm fulfills its goals.

Keywords. Estimation, Optimization problem, Random variable, scheduling Wireless sensor networks.

MSC. 90C26.

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1 Introduction

Wireless Sensor Network (WSN) is a network that has several geographically distributed sensor nodes that sense events in an environment. The sensor node's energy, computational power, and storage capacity are limited [13]. In the recent decade, WSN has developed in many applications and environments such as monitoring, healthcare, home automation, etc. [14]. Recently WSN also is used in distributed estimation, detection, and tracking.

In [14], a distributed estimation of an unknown target parameter by a set of sensed data is performed in a distributed sensor node environment. For simplicity, we call it the parameter. In this environment, each node directly or indirectly sends its data from the environment to a central FC. FC constructs the underlying physical phenomenon according to sensors data. There are much kinds of research about estimation in computer networks [1, 18] and in WSN [2, 34]. FC process received data and estimate the parameters. In many estimation types of research WSN [3, 19], they suppose that data sending is without any distortion. In [4], different distributed estimation algorithms reviewed [34, 26]. The decentralized estimation has been introduced in distributed control [6], target tracking [31] and data fusion [7]. In [35], an optimal power scheduling problem is proposed, is used in an inhomogeneous sensor network for a noise-corrupted deterministic signal. This algorithm quantizes the power level to minimize the total data sending energy consumption with Mean Square Error (MSE) performance. The [20] minimizes the estimated MSE with an optimal tradeoff between the number of alive nodes and the quantization bit rate. The paper [12] estimate a parameter variable in a bidimensional scenario in WSN. In this work, a mathematical framework is analyzed. In [8], the energy consumption performance is reviewed in WSN distributed estimation. This paper estimates the parameter with the Best Linear Unbiased Estimation (BLUE) method. The [8, 21, 8] use an optimization method to solve the problem. The [21] explicitly considers network lifetime techniques and estimation precision in distributed estimation problems. This method has an estimation model using confidence interval is explained, which uses the user's required precision as an input. This model estimates the parameter with a defined precision (based on user-defined precision using) using the confidence interval method. It also increases network lifetime. The scheduling consists of always actual, random on-off, adaptive on-off, and periodic on-off states [9]. The proposed algorithm is an adaptive on-off scheduling algorithm in which FC creates a scheduling method, and other nodes use that method [28]. We consider hierarchical (intra- and inter-cluster routing) sensor networks, and the scheduling and routing algorithms for each cluster independently. Therefore, using a centralized algorithm in each cluster (knowing the fact the cluster area is limited) is applicable and efficient. In each cluster, Intra-cluster routing sends sensor data to the cluster head (CH). The CH then communicates with other CH to route data to the sink node. Hear single-hop is applicable for routing between clusters. The proposed algorithm schedules nodes' activity with a nonlinear programming (NLP) method to send data from a node to FC by multi-hop routes. Some of the researchers perform scheduling nodes' tasks in the MAC layer [22, 32]. The [16] reviews different techniques using simulation in a many-to-one communication paradigm. It minimizes the number of time slots required

to complete a converge cast using a single frequency scheduling method. Some works are application-based. For example, in [27], a scheduling and routing algorithm guarantee the end-to-end delay. The [17] designs a lifetime-aware routing and coverage aware algorithm. The [10] is using a routing and scheduling algorithm in mesh sensor networks. This paper maximizes the lifetime of a WSN and guarantees the end-to-end delay. The other studies worked on routing and scheduling separately. They optimized just one [33].

In this paragraph, we emphasize the novelties of this paper. This paper emphasizes joint routine and scheduling algorithms. It is important to provide both routine and schedule at the same time because efficiency depends on it. Moreover, having multi-level routing helps us to provide more efficient energy consumption in nodes. We have used queuing theory to estimate delays in scheduling algorithms. In this paper, we proposed a hybrid hierarchical system with routing and scheduling for WSN. Regarding our last papers emphasized by the reviewer, it is worth mentioning that in this work, we have an optimization model efficiently adapted to the characteristics of the WSNs. By taking all similar papers in this field, including our last papers, into account, in the following, we have listed the main contributions:

- Adopted to tier hierarchical routing to the delay while the scheduling is optimized simultaneously.
- Unique confidence interval based error refinement based on HPD interval form.
- Considering both energy and delay in a mathematical model to have intra-cluster joint routing and scheduling.
- Having queue theory in delay provisioning.
- Having proposed a new solution for the proposed NLP regarding the running time.

We performed routing and scheduling algorithms jointly to achieve the highest efficiency. Also, the proposed algorithm is highly compatible with estimation process data, which is not well studied in literature before. Section 2 introduces the proposed algorithm to estimate a random variable parameter. In Section 3, proposed joint routing and scheduling algorithms are discussed in detail. The Sensors monitor the network and send their data to the CH, which estimates the parameter using the model described in Section 2. Our intra cluster routing and scheduling algorithm increases lifetime during routing from nodes to CH. The CH then estimates the required parameter and sends the results to the sink node. In Section 4, the performance of the proposed algorithm is evaluated against others. Section 5 has a conclusion.

2 Problem Statement

We define the hierarchical wireless sensor network topology consisting of different clusters. A cluster is shown in Figure 1. There are N sensor nodes and a CH in each

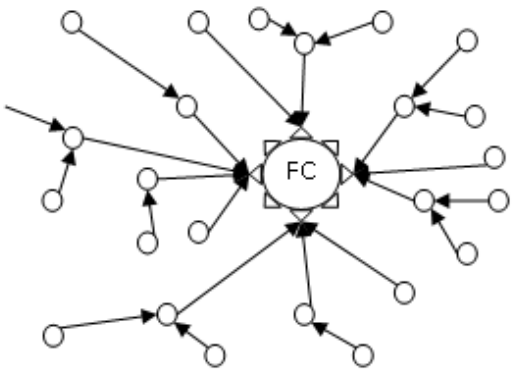

Figure 1: Sensor nodes as a cluster.

cluster to estimate the required parameter $\propto$. At first, each cluster node monitors the cluster environment and then sends events to the fusion center CH. At last, FC makes its estimation according to all the received data from cluster nodes.

When cluster nodes send data to FC, there are two main challenges:

1. how many packets should cluster nodes send to FC (section 3.1) to estimate the parameter $\propto$
2. in order to send cluster nodes data to FC, each node should select the best route and relay nodes (section 3.2).

During environment monitoring, the event data may damage by some additive noise:

$$x_{ki} = \propto + \varepsilon_{ki} \qquad k = 1, 2, \cdots, N, \quad i = 1, 2, \cdots, n_k. \tag{1}$$

In this relation x_{ki} is the i^{th} data of node k. Each node sends its data to FC Node k provide the n_k data size. N is the sensor network nodes. $\propto$ is the parameter monitored by the nodes. FC tries to estimate $\propto$ with the least possible error. The environment has noise, so a noise ε_{ki} is added to the parameter $\propto$, which is a random variable. Most of the applications are compatible with the random variable the parameter [25, 29]. The noise variables in the sensors,ε_{ki} , is independent, and the mean zero Gaussian random variable is $var(\varepsilon_{ki}) = \sigma_k^2$ $(k = 1, 2, \cdots, N)$. So we have $x_{ik} \sim N(\partial, \sigma_k^2)$, in which X_i have Gaussian distribution with mean ∂ and variance σ_k^2. In the estimation process (because of bandwidth and energy limitations of sensors), each node at first quantizes the event analog data locally y_k into a discrete message $m_k = Q_k(y_k)$ of length L_k bits [1] in which $Q_k(y_k)$ is quantization function.

In the scheduling algorithm, the cluster head determines all the parameters such as the number of sending data, the specified route, sending time, and the state of each node (ON or OFF). Our goal is to propose a scheduling algorithm to manage cluster nodes on or off activities and increase network lifetime and decrease estimation error under the desired bound.

2.1 Estimation process

Cluster nodes send data to FC to estimate the parameter $\propto$. The more the data, the more accuracy, and lower error. We use the Bayesian method to estimate $\propto$. We use the Normal distribution function with mean μ and variance τ^2 for the density function of $\propto$ random variable. Nowadays, most of the variables behavior is like Normal distribution function (such as temperature, humidity, etc.). They can be considered in future works. The density function of $\propto$ is as follow:

$$\pi(\propto)=\frac{1}{\sqrt{2\pi\tau^2}}e^{-\frac{1}{2\tau^2}(\propto-\mu)^2} \tag{2}$$

If x is an independent random variable with normal distribution function, it's joint density function is as follows:

$$f(x\mid\propto)=\prod_{i=1}^{n}f(x_i\mid\propto)=\prod_{i=1}^{n}(2\pi{\sigma_i}^2)^{-\frac{1}{2}}.e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\propto}{\sigma_i}\right)^2}. \tag{3}$$

To the equations (2) and (3), the joint density of $\propto$ and x is achieved using the equation (4):

$$\pi(\propto,x)=\frac{\prod_{i=1}^{n}\left(2\pi\sigma_i^2\right)^{-\frac{1}{2}}}{\sqrt{2\pi\tau^2}}e^{-\frac{1}{2}\left\{\sum_{i=1}^{n}\left(\frac{x_i-\propto}{\sigma_i}\right)^2+\frac{1}{\tau^2}(\propto-\mu)^2\right\}}. \tag{4}$$

By expanding the equation (4) we have:

$$\pi(\propto,\underline{x})=\frac{\prod_{i=1}^{n}\left(2\pi\sigma_i^2\right)^{-\frac{1}{2}}}{\sqrt{2\pi\tau^2}}e^{-\frac{1}{2}\left\{\sum_{i=1}^{n}\left(\frac{1}{\sigma_i^2}\left(x_i^2+\propto^2-2\propto x_i\right)\right)+\frac{1}{\tau^2}(\propto^2+\mu^2-2\mu\propto)\right\}}. \tag{5}$$

To achieve the posterior distribution function, the equation (5) can be changed as the following form:

$$\pi(\propto,\underline{x})=\frac{\prod_{i=1}^{n}\left(2\pi\sigma_i^2\right)^{-\frac{1}{2}}}{\sqrt{2\pi\tau^2}}e^{-\frac{1}{2}\left\{\propto^2\left(\sum_{i=1}^{n}\frac{1}{\sigma_i^2}+\frac{1}{\tau^2}\right)-2\propto\left(\sum_{i=1}^{n}\frac{x_i}{\sigma_i^2}+\frac{\mu}{\tau^2}\right)\right\}\times C}. \tag{6}$$

In the above equation, the terms which are not related to $\propto$ are represented by a constant coefficient C. This equation is a normal density function. Conditional density of $\propto$ is $\propto|\underline{x}\sim N(\acute{\mu},\acute{\sigma}^2)$, where $\acute{\sigma}^2$ and $\acute{\mu}$ are as follows:

$$\frac{1}{\acute{\sigma}^2}=\sum_{i=1}^{n}\frac{1}{\sigma_i^2}+\frac{1}{\tau^2}\quad=>\acute{\sigma}^2=1/\left(\sum_{i=1}^{n}\frac{1}{\sigma_i^2}+\frac{1}{\tau^2}\right), \tag{7a}$$

$$\frac{\acute{\mu}}{\acute{\sigma}^2}=\sum_{i=1}^{n}\frac{x_i}{\sigma_i^2}+\frac{\mu}{\tau^2}\quad=>\acute{\mu}=\left(\sum_{i=1}^{n}\frac{x_i}{\sigma_i^2}+\frac{\mu}{\tau^2}\right)/\left(\sum_{i=1}^{n}\frac{1}{\sigma_i^2}+\frac{1}{\tau^2}\right). \tag{7b}$$

The distribution function of $\propto|\underline{x}$ is normal and Bayesian estimator of the parameter $\propto$ is $\acute{\mu}$. So, the equation (7b) shows the final estimator [23].
We find the optimal sample number by calculating $(1-\alpha)\%$ with the quantity method

[24]. By considering $Q(\propto)=\frac{\propto-\acute{\mu}}{\acute{\sigma}}$ as pivotal quantity, the credible interval is given by the equation (8). Since our posterior density is unimodal, this credible interval is also an HPD interval form [24]:

$$P\left(\left|\frac{\propto-\acute{\mu}}{\acute{\sigma}}\right|<z_{\frac{\alpha}{2}}\right)>1-\alpha. \tag{8}$$

Here $z_{\frac{\alpha}{2}}$ is the $\left(\frac{\alpha}{2}\right)_{th}$ quantiles of standard normal distribution. By $(1-\alpha)\%$ credible interval and maximum acceptable error η, using the equation (9), data size is calculated:

$$z_{\frac{\alpha}{2}}\acute{\sigma}(n)<\eta. \tag{9}$$

Considering the equation (7a), the value of $\acute{\sigma}(n)$ is calculated as:

$$\acute{\sigma}(n)=\sqrt{1/\left(\sum_{i=1}^{n}\frac{1}{\sigma_i^2}+\frac{1}{\tau^2}\right)}. \tag{10}$$

Substituting the equation (10) in (9), we have:

$$z_{\frac{\alpha}{2}}{}^2\left\{1/\left(\sum_{i=1}^{n}\frac{1}{\sigma_i^2}+\frac{1}{\tau^2}\right)\right\}<\eta^2. \tag{11}$$

By extending the equation (11), we get:

$$\sum_{i=1}^{n}\frac{1}{\sigma_i^2}>\frac{z_{\frac{\alpha}{2}}{}^2}{\eta^2}-\frac{1}{\tau^2}. \tag{12}$$

Based on the equation (12), data size (n) is given by the equation (13):

$$n=\left\{min(n)\mid\sum_{i=1}^{n}\frac{1}{\sigma_i^2}>\frac{z_{\frac{\alpha}{2}}{}^2}{\eta^2}-\frac{1}{\tau^2}\right\}. \tag{13}$$

To simplify the formula of n we consider the worse situation as $\dot{\sigma}^2=max\left(\sigma_i^2\right)\mid i\in\{1,\ldots,N\}$. Therefore by replacing $\sum_{i=1}^{n}\frac{1}{\sigma_i^2}$ with $\frac{n}{\dot{\sigma}^2}$ in the equation (13), the optimal n can be calculated as follows:

$$n=\left\lceil\dot{\sigma}^2\left\{\frac{z_{\frac{\alpha}{2}}{}^2}{\eta^2}-\frac{1}{\tau^2}\right\}\right\rceil \tag{14}$$

As observable in the equation (14), data size is easily calculable when the user determines parameters α and η. The number of samples which are needed in order to achieve desired precision based on parameters α and η is known.

3 The Proposed Routing and Scheduling Program

There are many routing protocols to send cluster data to CH in WSNs. As the radio range in WSN is limited direct and indirect routing may be used, depending on network

conditions. We should know that direct transmission consumes more energy with less delay in comparison with indirect transmission, averagely.

Data size is determined based on the terms mentioned in section 2.1, to the application's required precision. The proposed algorithm regards two following challenges related to data transmission inside the cluster:

1. The route that cluster nodes can send data to FC.
2. Scheduling nodes activities and determining the way each node cooperates in the data gathering process.

In this section, methods for implementing the above issues are proposed.

3.1 Data transmission method inside the cluster

Wireless sensor network node consumes its energy for various reasons, to receive data, data collection, and data processing. Depending on the type of radio receiver, nodes have different energy consumption levels. However, in most WSNs, the same value has been considered for all network nodes. The volume of information that must be processed affects the node energy consumption. Similar to traditional networks, energy consumption due to data processing compared to other factors are negligible. Communication is the main factor of energy consumption in wireless sensor network nodes. Communication's energy consumption depends on several factors which are presented in the equation (15).

$$P_r/P_t = G_t G_r (C/4\pi d)^2. \tag{15}$$

We have studied different types of energy consumption model, and the model described in the equation (15) is the most suitable one. P_r is the signal strength received at the receiver side; P_t is the power of the signal at the transmitter side, G_t is transmitter antenna gain, G_r is receiver antenna gain strength, C is light speed, and d is the distance between sender and receiver. Note that the received signal should be greater than a specified threshold, thus based on the equation (16a), transmission energy consumption at the sender is calculated The G_r and G_T are determined according to node's characteristics, so they are constant (also C is constant). The value of d varies depending on the distance between the sender and receiver. We use the equation (16b) in order to determine transmission power. In (16b), we have: $\alpha \sim \left(G_t G_r (C/4\pi)^2\right)$.

$$E_t = F(G_t G_r (c/4\pi d)^2), \tag{16a}$$

$$E_t \sim \alpha/d^2 = F(\frac{1}{d^2}). \tag{16b}$$

Delay is also considered in the cost function. The Delay in the network is consists of propagation delay and queuing delay. The propagation delay in WSNs is minimal and can be ignored. So end-to-end delay only depends only on queuing Delay. We use $M/M/1/K$ for the nodes queue model. Modeling the delay by queuing theory is

acceptable when the traffic is almost constant. We consider monitoring applications that produce data with a constant rate. By considering the IEEE 802.15.4 as MAC layer protocol, it seems that the $M/M/1/K$ would be efficient enough for the problem [30].

In $M/M/1/K$, the input process is Poisson; the service rate is exponential, one server, and system capacity is K. Queuing delay is calculated as (17):

$$W = L/\acute{\lambda}. \tag{17}$$

The parameter L is the average queue length and $\acute{\lambda}$ is packets arrival rate. These are calculated with (18) [15] and (19) equations. In the equation (18), λ is the average input rate and ρ is the data density: $\rho = \lambda/\mu$,

$$L = (\rho/(1-\rho)) - \left((K+1)\rho^{K+1}/(1-\rho^{K+1})\right) = \dot{F}(\lambda, \mu, K), \tag{18}$$

$$\acute{\lambda} = \lambda(1-P_K), \tag{19}$$

P_K is, the probability of K packets in the queue and, is calculated as:

$$P_k = \begin{cases} ((1-\rho)\rho^k/(1-\rho^{K+1})) & (\rho \neq 1) \\ ((1/K+1)\rho^k) & (\rho = 1) \end{cases} \tag{20}$$

In direct routing, the cluster nodes transmit data to FC in one hop (some of the experts do not consider direct forwarding as a routing method, but in this text, we call direct forwarding as direct routing), and an indirect routing, cluster nodes send their data to FC in multihop routing. So the end-to-end delay will be the sum of intermediate nodes queuing delay. Figure 2, shows data routing inside the cluster.

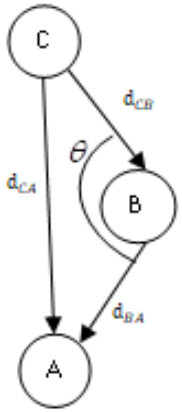

Figure 2: Data routing with three nodes.

In Figure 2, node A is the receiver, B is the relay, and C is the transmitter. Node C sends data to A directly on link CA with length D_{CA}. If node C sends its data to A using intermediate node B with Links CB and BA, it is indirect routing, Two scenarios are considered, direct (S1) and indirect (S2) transmission. The energy consumption for direct and indirect transmission is as the equations (21a) and (21b), respectively.

$$E_{s1} = F\left(d_{CA}^2\right) = \alpha d_{CA}^2, \tag{21a}$$
$$E_{s2} = F\left(d_{CB}^2\right) + F\left(d_{BA}^2\right) = \alpha\left(d_{CB}^2 + d_{BA}^2\right). \tag{21b}$$

In the above equation, function F exists in the equation (16a). The queuing model $M/M/1/K$ is used in all nodes A, B and C, so the end-to-end delay for direct and indirect routing is calculated as follow:

$$W_{s1} = L_A/\acute{\lambda}_A, \tag{22a}$$
$$W_{s2} = L_A/\acute{\lambda}_A + L_B/\acute{\lambda}_B, \tag{22b}$$

where L_A and L_B are average queue length of nodes A and B respectively. The $\acute{\lambda}_A$ and $\acute{\lambda}_B$ are actual input rate of nodes A and B respectively. L is as in the equation (18). We have:

$$W_{s1} = \dot{F}(\rho_A, K) = \dot{F}(\lambda_A, \mu_A, K), \tag{23a}$$
$$W_{s2} = \dot{F}(\rho_A, K) + \dot{F}(\rho_B, K) = \dot{F}(\lambda_A, \mu_A, K) + \dot{F}(\lambda_B, \mu_B, K). \tag{23b}$$

We suppose, all the network nodes are homogeneous, and are the same. In the equations (23a) and (23b), the μ_A and the μ_B are the service rate of nodes A and B. The μ_A and the μ_B are as characteristics of nodes when sensors are deployed in the.

$$Cost_{s1} = \beta_E . F\left(d_{CA}^2\right) + \beta_D . \dot{F}(\lambda_A, \mu_A, K), \tag{24a}$$
$$Cost_{s2} = \beta_E . \left(F\left(d_{CB}^2\right) + F\left(d_{BA}^2\right)\right) + \beta_D . \left(\dot{F}(\lambda_A, \mu_A, K) + \dot{F}(\lambda_B, \mu_B, K)\right). \tag{24b}$$

The β_E is the weighted cost of energy compared to delay. When β_E gets larger, it has more influence on the total cost. β_E is assigned by the user (25). Total cost is calculated by subtracting directly from indirect routing cost function:

$$Cost = Cost_{s1} - Cost_{s2}. \tag{25}$$

3.2 Forwarding scheduling

FC needs the cluster's topology to run the forwarding scheduling method. The topology is given to FC in CC matrix, In this matrix the CC_{ij} is the communication cost between nodes i and j. The initial value of CC matrix elements is as:

$$CC : [\ ]_{N\times N} \rightarrow CC_{ij} = \begin{cases} F(d_{ij}^2), & i,j \quad has\ direct\ path \\ \infty, & i,j \quad has\ indirect\ path \end{cases} \tag{26}$$

where N is the number of cluster nodes, d_{ij} is distance between nodes i and j. At the beginning of the process, the elements of CC matrix should be initialized. When nodes i and j are not located in their sending range, then CC_{ij} is considered infinite (the value will be replaced by the other values according to the selected least-cost routes). Otherwise they are in the radio range of each other. The CC matrix is obtained from

the network graph. The values of the CC matrix will be improved to the link cost. At the end of the algorithm, the CC matrix will have the best cost between each two sensor nodes.

The vector participation$_{N\times1}$ maintains a degree of each node participating in the routing process. It means how many routs a node i is in when data packets are sending to FC. Matrix Link$_{N\times N}$ have intermediate nodes located on the optimal path between every two nodes are placed in each other radio range. Matrix Links elements are a two-dimensional Vector$_{N\times2}$. If i and j are not in each other's radio range, Links$_{ij}$ is empty. The optimality of the paths is obtained from the cost function in section 3.1. The matrix CC Optimal value is obtained from Figure 3 pseudo code.

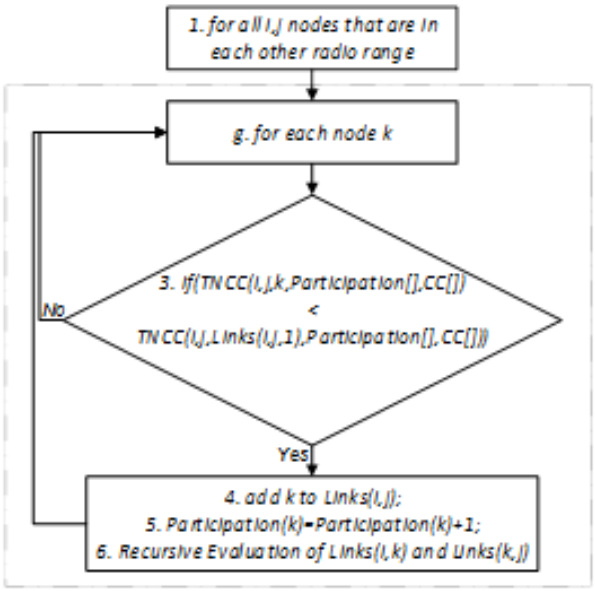

Figure 3: State diagram of determining matrix CC components.

If two nodes are in each other's radio range (each i and j), the algorithm searches the third node to send node's I data to node j, as the equation (20). Node k should be in the radio range of both nodes. TNCC function in Figure 3, compares the indirect $i-k-j$ and direct $i-j$ paths to find the best communication cost. If node k's communication cost is less than the current estimated cost, it will be selected as an intermediate node and is added to the Links$_{ij}$ vector, and the participation increases one. Then node k, as describes in the pseudo-code of Figure 3, links $i-k$ and $k-j$. After that, the algorithm is repeated. If the $i-k$ or K_J path can use another intermediate node, the node will be added to the array links.

If network conditions are modeled real, we can propose a good scheduling algorithm. Using the equation (16a), the energy consumption of each node is modeled. $EC_{N\times N\times h}$ matrix is considered to hold the node's energy consumption. EC matrix is created using the Path matrix 1's pseudo-code is used to specify the value of EC matrix elements.

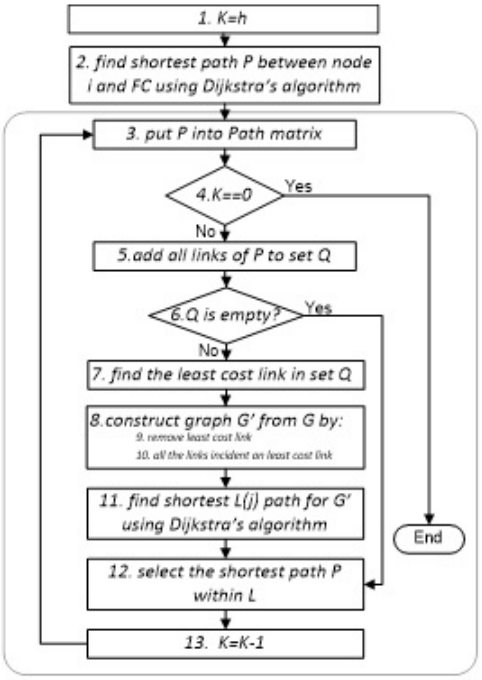

Figure 4: State diagram of finding h shortest path

Algorithm 1 The pseudo-code for EC matrix elements

```
for all i ∈ 1,...,N do
    for all j ∈ 1,...,N do
        for all k ∈ 1,...,N do
            if (Paths_{i,j,k+1<>0}) then
                EC_{i,j,k} = Econ(Paths_{i,j,k}, Paths_{i,j,k+1})
            end if
        end for
    end for
end for
```

Econ(i, j) function determines the required energy for transmitting a data unit from node i to j based on their relative distance, using the equation (16b). $EC_{i,j,k}$ element defines the amount of energy which is consumed by node j, where the k^{th} optimal path between node i and FC is selected. As mentioned before, we propose a scheduling algorithm and selecting an optimal path between each node and FC. Therefore matrix $SP_{R\times N\times N}$ is defined. In this matrix, N, the number of nodes in a cluster, and R is the number of algorithm estimation rounds. Each element in $SP_{i,j,k}$ shows the number of data samples sent by node j in i^{th} round in k^{th} optimal path. In the estimation process, the algorithm runs in independent rounds. Each round time duration is predefined. The network lifetime will be $R \times T$ for each round time, T, FC collects data from nodes in

each round, and estimates the parameter with desired precision. Equation (27) show the optimization problem proposed:

$$Min\ F=\gamma_1 N_1\Big(\sum_{j=1}^{R}\sum_{i=1}^{N}\sum_{k=1}^{N} SP(j,i,m).EC(j,i,k)\Big)+\gamma_2 N_2(Var(.))-\gamma_3 N_3(R) \tag{27}$$

$$\gamma_1+\gamma_2+\gamma_3=1 \tag{27a}$$

$$S.T:\ \forall k\in N,\ \sum_{j=1}^{R}\sum_{i=1}^{N}\sum_{m=1}^{N}(SP(j,i,m).EC(j,i,m))+\sum_{j=1}^{R}(E_{ac}.AP(j,k))<E_{pri} \tag{27b}$$

$$S.T:\ \forall j\in R,\ AP(j,k)=\left\lceil\sum_{i=1}^{N}\sum_{m=1}^{N}(SP(j,i,m).Path(i,k,m))\right\rceil \tag{27c}$$

$$S.T:\ \forall i\in R,\ \sum_{j=1}^{N}\sum_{k=1}^{N}SP(i,j,k)=\left\lceil\dot{\sigma}^2\left\{\frac{{z_{\frac{\alpha}{2}}}^2}{\eta^2}-\frac{1}{\tau^2}\right\}\right\rceil \tag{27d}$$

$$S.T:\ Var(.)=\sum_{q=1}^{N}\left\{\left(\sum_{j=1}^{R}\sum_{i=1}^{N}\sum_{m=1}^{N}(SP(j,i,m).EC(i,q,m))+\sum_{j=1}^{R}(E_{ac}.AP(j,q))\right)\right.$$
$$\left.-\left(\sum_{r=1}^{N}\sum_{j=1}^{R}\sum_{i=1}^{N}\sum_{m=1}^{N}(SP(j,i,m).EC(i,r,m))+\sum_{j=1}^{R}(E_{ac}.AP(j,r))/N\right)\right\}^2.$$

In the equation (27b) we see that nodes initial energy E_{pri} is bigger than node's energy consumption. A node consumes energy every time it is active and sends packet. So in a round if node is active it consumes a constant energy. If it is inactive, it consumes a negligible energy. The term $\sum_{j=1}^{R}\sum_{i=1}^{N}\sum_{m=1}^{N}(SP(j,i,m).EC(i,k,m))$ in the equation (27b), shows the energy consumption of a sending node. E_{ac} shows node's energy consumption in a round. Therefore term $\sum_{j=1}^{R}(E_{ac}.AP(j,k))$ shows total energy consumption of node during its lifetime. If a node does not send any data, it must be inactive. In the equation (27c) the relation between matrixes P, SP and path are shown.

The proposed scheduling and routing algorithm uses an NLP to find the best paths and node activity. The object and subjects of the proposed NLP are not very complicated. We test it using Matlab[1], and the answers and the response time is acceptable. The proposed NLP is acceptable to execute in a sensor node. Furthermore, the proposed scheduling and routing (including the proposed NLP) are performed once, so its complexity is negligible. Each CH solves the NLP once (the output will be the scheduling program and the routing paths), and informs the sensor member nodes, so the overhead is acceptable. So, our algorithm is efficient because the parameter distribution function is known. This function is compatible with estimating the parameter with desired precision. All parts of the proposed algorithm are fully adjusted to the application requirements.

[1] www.matlab.com

3.2.1 Matrix SP

In the equation (27), the best values of matrix SP elements are obtained. In this form, solving optimization problems is so hard, because the parameter R is on summation bound. We should replace parameter R with $\dot{R}$ in the equation (27) to solve the problem. The new objective function will be as:

$$Min\ F = \gamma_1 N_1 (\sum_{j=1}^{\dot{R}} \sum_{i=1}^{N} \sum_{k=1}^{N} SP(j,i,m).EC(j,i,k)) + \gamma_2 N_2 (Var(.)) - \gamma_3 N_3(R). \quad (28)$$

This optimization problem has just one difference with the equation (27). The parameter R in the equation (27) is unknown, but in (28), $\dot{R}$ is known. In the following the proposed method, we defined parameter $\dot{R}$ and solved the equation (27). In line 1 of 2, the parameter $R(0 < R < RM)$ (here it is known as $\dot{R}$) is introduced.

Algorithm 2 Proposed optimal R algorithm

1: **for** $\dot{R} \in (0 - R_M)$ **do**
2: $LB = 0, UB = R_M$
3: **end for**
4: **for all** $i \in 1, ..., \log_2 R_M$ **do**
5: $\dot{R} = (LB + UB) \div 2$
6: solve the equation(22b) using $\dot{R}$
7: **end for**
8: **if** Equation(22b) is solved successfully **then**
9: $LB = \dot{R}$
10: **else**
11: $UB = \dot{R}$
12: **end if**
13: **return** $\dot{R}$

In Section 3.2.2, we find R_M value. Parameter R is the network lifetime. if $R = 0$, it means that the network cannot perform even one round. In line 4, the problem is solved successfully. There are two points here, 1) if $\dot{R} >$ optimal (R), then the optimization problem (Equation (27)) has no answer. This is because of the contradiction between the equations (27b) and (27d). If the optimization problem is solved for the current value of $\dot{R}$, we can accept the solution.

3.2.2 Calculating R_M

To calculate R_M , we have considered the over the optimal situation which, is not practical in reality. Therefore a network with the following conditions has been considered:

1. it consists of only one node (called selected node)
2. initial energy of selected node is E_T

3. the selected node consumes E_L energy unit due to send each message to FC. Considering that only one node exists in the network, the equation (27d) is rewritten (29). Parameters E_L and E_T are calculated using the equations (30) and (31), respectively.

$$S.T: \forall i \in R, \quad \sum_{j=1}^{N}\sum_{k=1}^{N} SP(i,j,k) = \left\lceil \sigma^2 \left\{ \frac{z_{\frac{\alpha}{2}}{}^2}{\eta^2} - \frac{1}{\tau^2} \right\} \right\rceil, \tag{29}$$

$$E_L = min(E_{i\in(1:N)}), \tag{30}$$

$$E_T = N \times E_{pri}. \tag{31}$$

In the definition of $\dot{\sigma}(\dot{\sigma}^2 = max\left(\sigma_i^2\right) \mid i \in \{1, \ldots, N\})$, we have: $\sigma^2 < \dot{\sigma}^2$. In the equation (30), E_L is the minimum energy consumption of all nodes in a cluster. In the equation (31), parameter E_T is the initial energy of all the cluster nodes. With equations (29),(30) and(31), R_M is obtined as:

$$R_M = \left\lceil E_L / \left(\left\lceil \sigma^2 \left\{ \frac{z_{\frac{\alpha}{2}}{}^2}{\eta^2} - \frac{1}{\tau^2} \right\} \right\rceil . E_T \right) \right\rceil. \tag{32}$$

4 Performance Evaluation

We used MATLAB and OPNET software in investigating the performance of the proposed algorithm[1]. Calculations are implemented using MATLAB software, and network simulations are performed using OPNET software. Both software are based on C compiler, and we connected them. We call the MATLAB engine in OPNET environment. The main contribution of the proposed algorithm is to estimate the parameter based on user-defined precision. Also, joint routing and scheduling algorithms are provided to send the estimation output to FC. The proposed algorithm is fully adjusted to the proposed estimation model requirements. As it is presented in the rest of this section, by considering two heuristic algorithms, the proposed algorithm achieves its goals. We used heuristic algorithms because there is no previous work that performs the estimation and data gathering similar to our proposed algorithm (this model of evaluation is expected in this field [11, 5]).

We evaluate our algorithm efficiency with the following similar algorithms:

1. The proposed algorithm with single-hop routing, called $M1$.

2. The proposed algorithm with one route between each node and FC, called $M2$.

In this section, we will call our proposed algorithm, $M3$. Figure 5, shows the relation between precision parameters and data size.

[1] www.Opnet.com

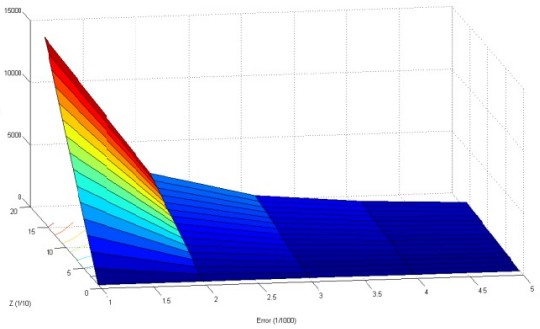

Figure 5: The relationship between data size and precision parameters.

The data size is calculated by the equation (14). The parameter sσ̇, η, $z_{\frac{\alpha}{2}}$, and τ has an effect on data size. The values of σ̇ and τ are determined based on network conditions. However, parameters $z_{\frac{\alpha}{2}}$ and η determine estimation precision, which is directly specified by the user. The X-axis in Figure 5, shows the acceptable error limitations. The scale of the x-axis is 1/1000. Y-axis shows the degree of certainty. Y-axis maps $z_{\frac{\alpha}{2}}$. The z-axis presents data size (n). It is evident in Figure 5 that, for more precision, smaller acceptable error limitations and a high sample size are required.

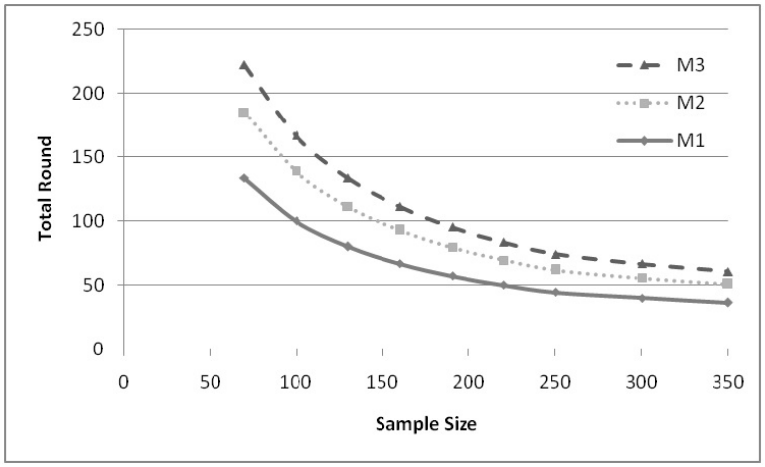

Figure 6: Total performed rounds versus different sample size

In Figure 6, we see the total performed rounds with different required sample sizes. In each round, if nodes don't have enough energy to run the estimation, they cannot finish that round. When network nodes send required data samples to FC, the estimation is complete. So, in Figure 6, it's clear that the lifetime of $M3$ is more than the other two algorithms because it can complete more rounds. The efficiency of $M3$ is more distinct in lower data size. The horizontal axis in Figure 6 shows the data size that is determined based on σ̇, η, $z_{\frac{\alpha}{2}}$ and τ values.

Figure 7 shows the average packet transmission energy consumption for all algorithms. The X-axis is the number of network nodes, and the y-axis is the average energy consumption. Algorithms $M1$, $M2$, and $M3$ are compared with each other.

Since $M1$ uses direct routing, average energy remains constant. As shown in Figure 7, the average energy consumption of $M2$ is less than $M3$; this happens, because $M3$ has a fair and more efficient routing algorithm. As it is clear, if the network consumes energy at a fair, the result will be efficient. Therefore algorithms in which some nodes corporate in data sending more than other nodes. They lose energy faster. In some situations, $M3$ may use non-optimal routes to provide fairness in the node's energy consumption. However, $M2$ always selects the least cost route. Therefore, as is shown in Figure 8, the number of estimation rounds for $M3$ is more than two other algorithms.

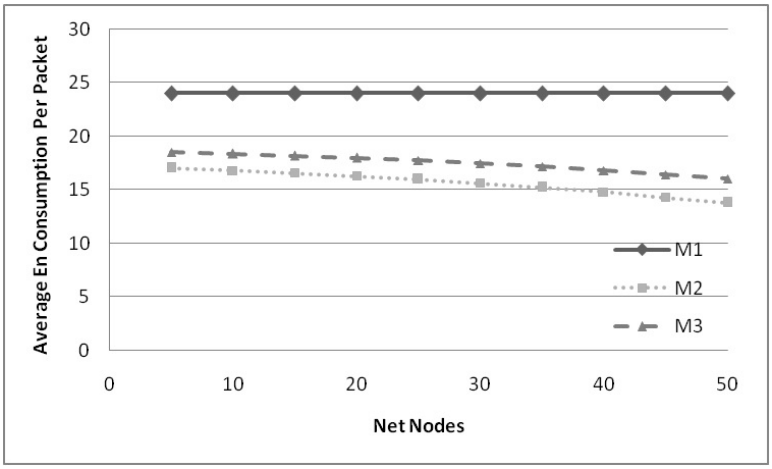

Figure 7: The average packet energy consumption versus the number of nodes.

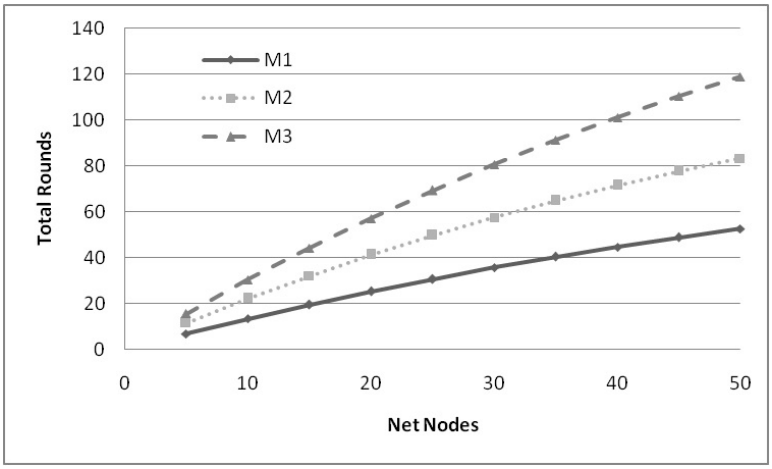

Figure 8: The number of performed estimation rounds versus the number of nodes.

As mentioned earlier, the main reason behind M3 efficiency in comparison with $M2$ is providing fairness. Like the $M2$ algorithm, $M3$ considers multiple least cost routes instead of one route. The Variance is obtained as follows:

$$Var = \sum_{i=1}^{N} (RE_i - AE)^2 \tag{33}$$

where RE_i is the remained energy of i^{th} node and, AE is all network node's average remained energy.

In Figure 9, energy variance of nodes for $M1$, $M2$ and, $M3$ algorithms are shown. Variance shows fairness for each algorithm. As can be seen in Figure 9, the $M3$ algorithm has a lower Variance. It is near zero. In this regard, $M1$ has the least efficiency because it uses direct forwarding.

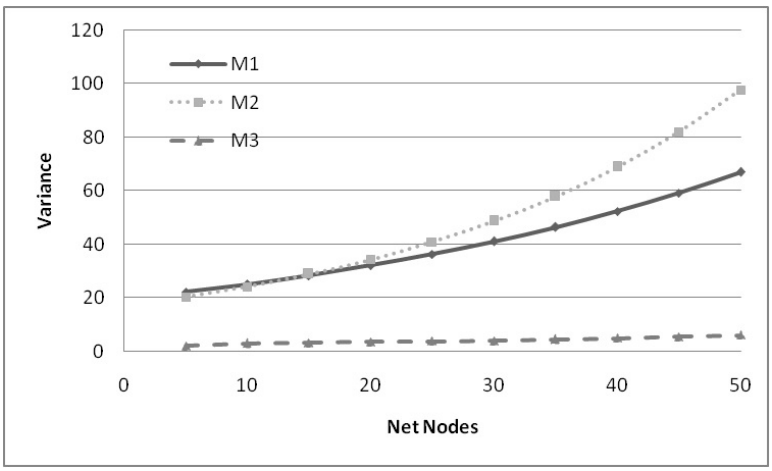

Figure 9: The variance of the remained network energy versus the number of nodes.

Figure 10 shows the packet's average end-to-end delay. Due to short distance limitations in sensor networks, we ignore propagation delay. Therefore, a delay only depends on the queuing delay. Note that delay is computed by the application layer.

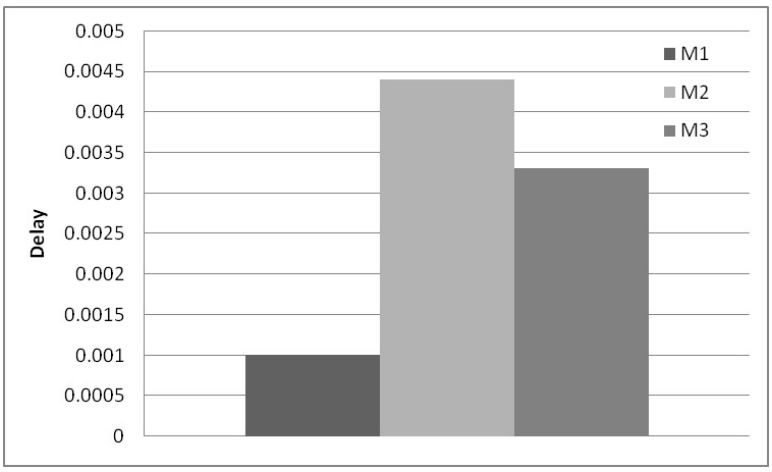

Figure 10: End-to-end delay in three algorithms.

Figure 10 shows that the M1 algorithm has a lower delay than the other algorithms, because in this algorithm, nodes send their data to the FC node directly. Also delay of the $M3$ algorithm is less than $M2$, because the number of intermediate nodes in this algorithm is lower than $M2$ because of energy consumption fairness.

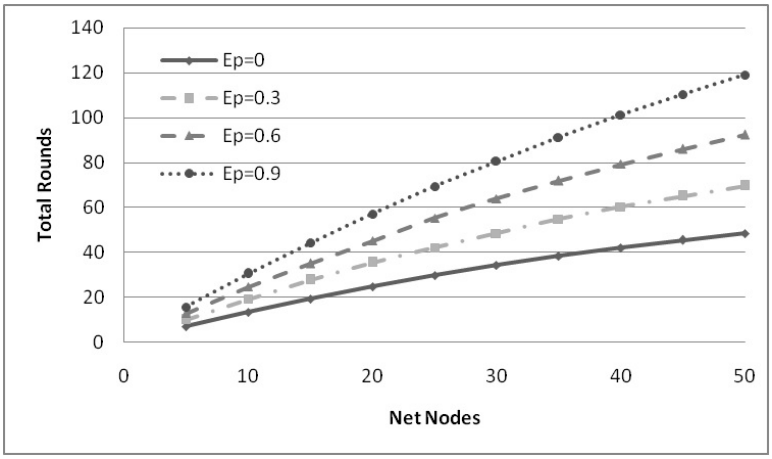

Figure 11: Total rounds versus the number of nodes.

Figure 11 shows the total round versus the number of nodes for $M3$. $E_p(\beta_E)$ parameter, as introduced in equation (14), determines energy priority. When E_p gets closer to 1, the $M3$ algorithm gets more energy-efficient, and the network lifetime increases.

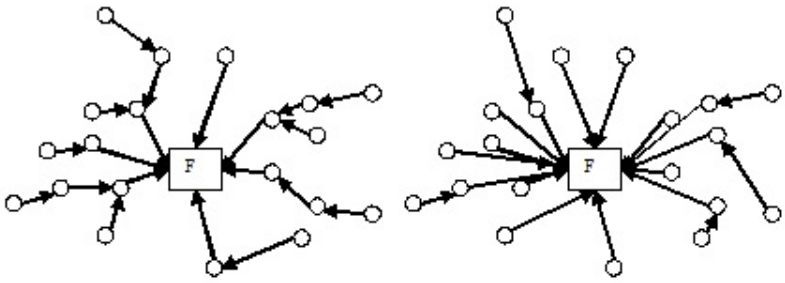

Figure 12: Intra-cluster routing tree, A-left) $\beta_E = 1$, B-right) $\beta_E = 0.1$

In Figure 12, intra cluster routing tree is presented to different values of $E_p(\beta_E)$. 14-A presents the routing tree when $\beta_E = 1$ and 14-B presents the routing tree when $\beta_E = 0.1$.

5 Conclusion

The proposed algorithm estimates the parameter with desired precision and increases network lifetime. This algorithm consists of a routing and scheduling algorithm to send data to FC and estimate a parameter in WSN. The network lifetime is calculated by the

number of successful estimation rounds. The network will be non-functional if nodes don't have sufficient energy to provide enough samples for FC. To achieve the goals, the scheduler uses a nonlinear optimization problem (NPL). Based on the required accuracy of estimation, the proposed NLP not only determines the routes (multihop routing) but also determines the state of each sensor node (active or inactive). Regarding the environment characteristics, the parameter is considered a random variable, which is estimated using a Bayesian confidence interval based on the user's desired precision. The cost function of the multihop routing considers both delay and energy parameters. The proposed algorithm was compared with the other two algorithms that showed more efficiency. Without loss of generality, calculations are performed central, and links are considered loss-less. Finally, in future works, we will consider a fully distributed method to solve the optimization problem based on the link's error rate.

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Persian Abstracts

شرایط لازم بهینگی برای مسائل زمان-پیوسته ناهموار با استفاده از محدب‌کننده‌ها

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چکیده

در این مطالعه، شرایط لازم بهینگی از نوع کارش-کان-تاکر را برای یک مساله‌ی بهینه‌سازی زمان-پیوسته‌ی ناهموار تعمیم می‌دهیم. ابزار اصلی ما برای بدست آوردن شرایط بهینگی، استفاده از مفهوم محدب‌کننده است. ما با استفاده از این مفهوم ابتدا توصیف قیدی مانگاساریان-فروموویتز را برای این دسته از مسائل تعمیم می‌دهیم، سپس شرایط لازم بهینگی را تحت فرض‌های ضعیف بدست می‌آوریم. در واقع در این مقاله، تابع هدف و توابع قیود را ناهموار و غیرمحدب در نظر می‌گیریم.

کلمات کلیدی

مسائل زمان-پیوسته، شرایط بهینگی، محدب‌کننده نیم-منظم بالایی، آنالیزناهموار.

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چکیده

مسائل برنامه‌ریزی خطی دارای پارامترهای دقیق هستند. در دنیای واقعی، ما با شرایطی روبرو هستیم که اطلاعات دقیق و کامل در دسترس نیست. در این شرایط می‌توان از رویکردهای عدم قطعیت مانند فازی و تصادفی برای مقابله با عدم اطمینان در زندگی واقعی استفاده کرد. اگر تعداد متخصصان و سطح تجربه به قدری کم باشد که استخراج توابع عضویت غیرممکن باشد یا تعداد نمونه ها کم باشد، نمی‌توان از نظریه‌های فازی و تصادفی استفاده کرد. برای حل این مشکل، نظریه سیستم خاکستری ارائه شد. در این مقاله ، یک مساله برنامه‌ریزی خطی در یک محیط خاکستری با منابعی به صورت اعداد خاکستری بازه‌ای در نظر گرفته شده است. بیشتر روش‌های پیشنهادی برای حل مسائل برنامه‌ریزی خطی خاکستری بر مبنای تبدیل آن به یک مساله برنامه‌ریزی خطی معمولی می‌باشند. با این حال، ما به دنبال حل مستقیم مساله و بدون تبدیل آن به یک مساله برنامه‌ریزی خطی استاندارد به منظور حفظ عدم قطعیت داده‌های ورودی در جواب‌های بدست آمده نهایی هستیم. برای این منظور، ما یک روش، بر اساس دوال مساله برنامه‌ریزی خطی با منابع خاکستری ارائه می‌دهیم. روش پیشنهادی ساده‌تر از روش‌های قبلی بوده و از پیچیدگی کمتری نسبت به آنها برخوردار است. تأکید می‌شود که روش ارائه شده برای شرایط واقعی و عملی در مسائل مدیریت و برنامه‌ریزی می تواند مفید باشد. بنابراین، کارایی روش پیشنهادی با ارائه چند مثال در شرایط مختلف نشان داده شده است.

کلمات کلیدی

عدد خاکستری، برنامه‌ریزی خطی خاکستری، نظریه دوگان، عدم قطعیت.

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چکیده

در این مقاله متری برای اندازه‌گیری فاصله بین دو عدد فازی به صورت یک عدد فازی پیشنهاد می‌شود. برای این منظور، فاصله فازی به صورت یک معیار وزنی از α-مقدارهای اعداد فازی تعریف می‌شود و خاصیت‌های اساسی آن در محیط مورد بررسی قرار می‌گیرد. نحوه محاسبه فاصله فازی دو عدد فازی با یک مثال توضیح داده می‌شود. همچنین فاصله پیشنهادی با دیگر انواع مشابه از نظر ساختار مورد مقایسه قرار می‌گیرد.

کلمات کلیدی

عدد فازی، α-مقادیر، متر فازی، پایایی.

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چکیده

لجستیک امدادی و زنجیره تأمین بشردوستانه در ادبیات دانشگاهی به فرایند برنامه‌ریزی، اجرا و کنترل اثربخش جریان هزینه‌ها و اطلاعات و ذخیره‌سازی کالاها و مواد موردنیاز از نقطه مبدأ تا مصرف اطلاق می‌شود به‌گونه‌ای هدف اصلی آن کاهش و تسکین درد و رنج مردم حادثه‌دیده می‌باشد. در این مقاله به ارائه یک مدل چندهدفه برای مساله مکانیابی-مسیریابی چند دوره‌ای با در نظر گرفتن تخلیه مصدومین و افراد بی‌خانمان و مسیرهای فازی در لجستیک امداد پرداخته شد. ابتدا یک مدل چندهدفه غیرقطعی از مساله تحت پارامترهای غیرقطعی تقاضا، زمان و ظرفیت حمل‌ونقل طراحی و سپس با استفاده از روش برنامه‌ریزی فازی به کنترل پارامترهای غیرقطعی مساله پرداخته شد. با توجه به NP-سخت بودن مساله و عدم توانایی نرم‌افزار GAMS برای حل مدل در اندازه‌های بزرگ‌تر از الگوریتم‌های فراابتکاری NSGA-II و MOPSO برای حل مساله استفاده شد.

کلمات کلیدی

لجستیک امدادی، برنامه‌ریزی فازی، عدم قطعیت، الگوریتم‌های فراابتکاری.

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چکیده

در این مقاله، یک روش جدید برای حل معادله دیفرانسیل تصادفی ارائه می‌دهیم و معادله وازیچک را حل می‌کنیم. در ابتدا، نحوی محاسبه انتگرال‌های چندگانه ایتو بر اساس ساختار حرکت براونی ارائه شده به طوری که خطای محاسباتی در آن به حداقل می‌رسد. سپس، موجک‌های براونی یک بعدی و چند بندی بر اساس حرکت براونی معرفی می‌شوند. پس از آن، سیستم معادلات خطی و غیرخطی بر اساس این موجک‌های براونی بدست می‌آید که با حل این سیستم، معادله وازیچک حل می‌شود. در آخر چند مثال عددی آورده شده است.

کلمات کلیدی

معادله دیفرانسیل تصادفی، معادله وازیچک، حرکت براوانی، موجک براوانی، انتگرال ایتو.

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چکیده

در این مقاله تخمین پارامتر هدف با تابع توزیع مشخص با استفاده از مشاهدات گره حسگر انجام می‌شود. در شبکه‌های حسگر گره‌ها داده‌های واقعی را از محیط دریافت و آنها را به یک مرکز ارسال می‌کنند. این مرکز داده‌ها را جمع‌آوری و پارامتر مشاهده شده را با دقت تعریف شده کاربرد تخمین می‌زند. هدف اصلی الگوریتم پیشنهادی افزایش طول عمر شبکه در کنار بهبود کارایی فرایند تخمین است. برای نیل به این هدف الگوریتم پیشنهادی فعالیت گره را زمان‌بندی کرده و یک مسیر چند پرشی بین گره‌ها و مرکز مشخص می‌کند. نتایج شبیه‌سازی نشان می‌دهد که الگوریتم پیشنهادی، اهداف مورد انتظار را برآورده کرده است.

کلمات کلیدی

تخمین، مساله بهینه‌سازی، متغیر تصادفی، زمان‌بندی، شبکه‌های حسگر بی‌سیم.

فرم اشتراک

علاقه‌مندان به اشتراک نشریه

Control and Optimization in Applied Mathematics-COAM

می توانند فرم ذیل را تکمیل نمایند و به همراه فیش بانکی به مبلغ ۵۰۰۰۰ ریال (پنجاه هزار ریال) به شماره حساب :۲۱۷۸۶۰۹۰۰۱۰۰۷ و معادل شبای متمرکز ۰۷ ۸۶۰۹ ۰۲۱۷ ۰۰۰۰ ۰۱۷۰ IR ۴۲ ، نزد بانک ملی ایران شعبه بنفشه، کد: ۱۵۰۸ به دبیرخانه مجله ارسال دارند تا مجله برای آنان فرستاده شود.

نام: نام خانوادگی:

نام موسسه یا مرکز: رشته تحصیلی:

آخرین مدرک تحصیلی: شماره اشتراک:

اشتراک از شماره..... تا شماره ..... و تعداد مورد نیاز از هر شماره ..... نسخه

نشانی کامل پستی:

کدپستی: تلفن همراه:

تلفن تماس: پست الکترونیکی:

دورنگار: امضاء:

تاریخ: