Editorial Policy

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Journal of “Control and Optimization in Applied Mathematics (COAM)” is published twice a year (Spring-Autumn) by Payame Noor University (PNU). The COAM endeavors to publish significant research of broad interests in applied mathematics in the fields of Control and Optimization. For more information, one can see the Aims and Scope at the journal’s website.

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In the name of God

Control and Optimization in Applied Mathematics

(COAM)

Volume 3, Number 2, Autumn-Winter 2018

The journal of "Control and Optimization in Applied Mathematics-COAM" is authorized under the registration No. 91/35952 dated 1391/11/30 A.H.S. (Feb 18, 2013) by the Ministry of Culture and Islamic Guidance. This academic journal is indexed by "MSRT", Iranian Ministry of Science Research and Technology of IRAN (reference No.:3/155605 date:29 Sep 2018), Elmi-Pajooheshi in persian. This journal endeavours to publish significant research of board interests in applied mathematics in the fields of Control and Optimization. The COAM is published semi-annual, one volume is published each year, and each volume consists of two issues.

Period of publication:	Semi-annual
Subject :	Mathematics
p-ISSN:	2383-3130
e-ISSN:	2538-5615
Language:	English
Fifth Issue:	Autumn-Winter 2018
Owner :	Payame Noor University
Phone:	+9851-38683900
Fax:	+9851-38683001
Editorial Office:	Moallem BLV., Ave. 71, Payame Noor University, Mashhad, Iran.
P.O. Box:	91735-433
Website:	http://mathco.journals.pnu.ac.ir
Number of Offprints:	25 copies
E-mail Address:	mathco@pnu.ac.ir
Account Number:	2178609001007 Melli Bank (Branch of Banafshe)
Sheba Number:	IR 42 0170 0000 0217 8609 07
LaTeX Designer:	Sayyed Mahdi Moosavi

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This publication ethics is a commitment which draws up some moral limitations and responsibilities of research journals. The text is adapted according to the "Standard Ethics", approved by the Ministry of Science, Research and Technology, and the publication principles of Committee on Publication Ethics (COPE).

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References

[1] "Standard Ethics", approved by Vice-Presidency for Research & Technology, the Ministry of Science, Research and Technology.

[2] Committee on Publication Ethics, COPE Code of Conduct, www.publicationethics.org.

Control and Optimization in Applied Mathematics (COAM)

This journal is published under the auspices of the Payame Noor University (PNU).

*Full Professor
**Associated Professor

Control and Optimization in Applied Mathematics (COAM)

Editor in Chief's Letter

It would be our great honor to have you as the readers of Journal of "Control and Optimization in Applied Mathematics (COAM)". The present journal is published and supported by Payame Noor University (PNU) as a semi-annual journal. Our main objective is to facilitate scientific regional and global discussions and collaborations between specialists in different fields of applied mathematics, especially in the fields of control and optimization. We hope that scholars and experts of different fields of applied mathematics find our scientific journal a platform for international communications of insight and knowledge. To assure the respectful subscribers about high quality of the journal, each article is reviewed by subject-qualified referees, the same as any other well-known international journal of applied mathematics.We believe that by publishing high quality and creative researches, we will observe more collaborations with our journal. We kindly invite all applied mathematicians especially in the fields of control and optimization, to join us by submitting their original works to the Journal of "Control and Optimization in Applied Mathematics". I want to thank the respectful colleagues of COAM, as well as referees, reviewers, and editors for their kind dedication and vision.

Contents

Control and Optimization in Applied Mathematics (COAM)
DOI. 10.30473/coam.2020.50781.1133
Vol. 3, No. 2, Autumn-Winter 2018(1-12), ©2016 Payame Noor University, Iran

Quasi-Gap and Gap Functions for Non-Smooth Multi-Objective Semi-Infinite Optimization Problems

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Received: January 8, 2020; **Accepted:** April 2, 2020.

Abstract. In this paper, we introduce and study some new single-valued gap functions for non-differentiable semi-infinite multiobjective optimization problems with locally Lipschitz data. Since one of the fundamental properties of gap function for optimization problems is its abilities in characterizing the solutions of the problem in question, then the essential properties of the newly introduced gap functions are established. All results are given in terms of the Clarke subdifferential.

Keywords. Multiobjective optimization, Semi-Infinite Programming, Gap function, Clarke subdifferential

MSC. 90C34; 90C40; 90C26.

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1 Introduction

The notion of gap function for mathematical programming problems has been studied in various publications. This concept was first defined by Hearn in [7] for the scalar value convex optimization problems, and was then introduced for variational inequality problem in [1].

For multi-objective optimization problems with smooth data, the gap function has been presented in [4] as a set-valued function. Also, two kinds of set-valued gap functions are introduced for smooth and non-smooth multiobjective optimizations in [14]. Since the initial calculations of set-valued functions are faced with special problems, working with these gap functions is very difficult. Recently, Caristi *et al.* [4] introduced some single-valued gap functions, with complex structures, for multi-objective optimization problems.

All previously mentioned papers considered the (multiobjective) optimization problems with the finite number of constraints. Kanzi and Soleymani-Damaneh [10] studied the concept of gap function for optimization problems with the infinite number of quasi-convex constraints, i.e., quasi-convex semi-infinite problems. Also, the concept of gap function extended to linear semi-infinite multiobjective optimization in [11], and quasi-variational inequality problems in [13].

The purpose of this article is to introduce several scalar-valued gap functions, with simple structures, for semi-infinite multi-objective optimization problems with locally Lipschitz functions. In fact, the purpose of the present paper is to give a generalization of sources listed above. The paper mainly deals with constrained optimization problems formulated as

$$(P) \quad \begin{cases} \text{minimize } f(x) := \big(f_1(x), \ldots, f_p\,(x)\,\big) \\ \text{subject to } \ g_\alpha\,(x) \leq 0 \ \text{ with } \alpha \in A, \end{cases}$$

where $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ for $i \in \Delta := \{1, \ldots, p\}$ and $g_\alpha : \mathbb{R}^n \to \mathbb{R}$ for $\alpha \in A$ are (not necessary differentiable) locally Lipschitz functions, and the index set $A \neq \emptyset$ is arbitrary.

It is worth mentioning that Mastroeni [12] presented a descent method for solving the variational inequalities and optimization problems (under differentiability) based on gap function algorithms. Also, some applications of gap functions in iteration algorithms, proper efficiency, and scalarization of multiobjective optimization can be studied in [4, Section 5].

2 Notations and Preliminaries

In this section, we present definitions and auxiliary results that will be needed in the rest of the paper.

Let $\mathbb{R}^m$ be the $m-$dimensional Euclidean space. Denote by 0_m and $\mathbb{R}^m_+$ the zero vector (i.e., $(\overbrace{0,\ldots,0}^{m \text{ times}})$) and the nonnegative orthant of $\mathbb{R}^m$, respectively. Also, the open ball with center $a \in \mathbb{R}^m$ and radius $\varepsilon > 0$ is denoted by $\mathbb{B}_\varepsilon(a)$. The order and weak order in $\mathbb{R}^m$ can respectively be defined by :

$$\left(a^1,...,a^m\right) \leq \left(b^1,...,b^m\right) \iff \begin{cases} a^i \leq b^i, & \forall i = 1,\ldots,m, \\ a^l < b^l, & \exists l \in \{1,\ldots,m\}, \end{cases}$$
$$\left(a^1,...,a^m\right) < \left(b^1,...,b^m\right) \iff a^i < b^i, \quad \forall i = 1,\ldots,m.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of φ at $\hat{x} \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$, and the Clarke subdifferential of φ at $\hat{x}$ introduced in [8] are respectively given by

$$\varphi^0(\hat{x}; v) := \limsup_{y \to \hat{x},\ t \downarrow 0} \frac{\varphi(y+tv) - \varphi(y)}{t},$$

$$\partial_c\varphi(\hat{x}) := \left\{\xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \leq \varphi^0(\hat{x}; v) \quad \text{for all } v \in \mathbb{R}^n\right\}.$$

The Clarke subdifferential is a natural generalization of the derivative since it is known that when function φ is continuously differentiable at $\hat{x}$, then $\partial_c\varphi(\hat{x}) = \{\nabla\varphi(\hat{x})\}$.

Theorem 1. *(*Lebourg mean-value [8]*)* Let $x, y \in \mathbb{R}^n$, and suppose that φ is a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, there exists a point u in the open line segment (x, y), such that

$$\varphi(y) - \varphi(x) \in \langle \partial_c\varphi(u), y - x\rangle .$$

Definition 1. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. We say that φ is $c-$quasiconvex (i.e., Clarke quasiconvex) at $\hat{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$

$$\varphi(x) \leq \varphi(\hat{x}) \Longrightarrow \left\langle \xi, x - \hat{x}\right\rangle \leq 0 \quad \forall \xi \in \partial_c\varphi(\hat{x}).$$

3 Main Results

As a starting point of this section, we introduce the available set of (P) and the set of active indices a possible point x_0 as follows:

$$S := \{x \in \mathbb{R}^n \mid g_\alpha(x) \leq 0, \quad \forall \alpha \in A\},$$
$$A(x_0) := \{\alpha \in A \mid g_\alpha(x_0) = 0\}.$$

A given point $x_0 \in S$ is said to be an efficient (resp. weakly efficient) solution for (P) if there is no $x \in S$ satisfies $f(x) \leq f(x_0)$ (resp. $f(x) < f(x_0)$). The set of all efficient solutions and weakly efficient solutions of (P) are denoted by E and W, respectively.

For each $x_0 \in S$, let:

$$\widehat{\partial}_c f_i(x_0) := \partial_c f_i(x_0) \setminus \{0_n\}, \qquad \forall i \in \Delta,$$
$$\widehat{\partial}_c f(x_0) := \widehat{\partial}_c f_1(x_0) \times \ldots \times \widehat{\partial}_c f_p(x_0) \subseteq (\mathbb{R}^n)^p,$$
$$\partial_c^\sharp f(x_0) := \partial_c f(x_0) \setminus \{0_{np}\} = \Big(\partial_c f_1(x_0) \times \ldots \times \partial_c f_p(x_0)\Big) \setminus \{0_{np}\}.$$

It is easy to see that

$$\widehat{\partial}_c f(x_0) = \Big\{(\xi_1, \ldots, \xi_p) \in \partial_c f(x_0) \mid \xi_i \neq 0_n \text{ for all } i \in \Delta\Big\},$$
$$\partial_c^\sharp f(x_0) = \Big\{(\xi_1, \ldots, \xi_p) \in \partial_c f(x_0) \mid \xi_i \neq 0_n \text{ for some } i \in \Delta\Big\},$$
$$\widehat{\partial}_c f(x_0) \subseteq \partial_c^\sharp f(x_0) \subseteq \partial_c f(x_0).$$

First, we introduce a quasi-gap function for (P).

Definition 2. For each $(x, y, z) \in S \times S \times \mathbb{R}^n$ and $\xi := (\xi_1, \ldots, \xi_p) \in \partial_c f(z)$, the quasi-gap function $\varphi_y(x, z, \xi)$ is defined as:

$$\varphi_y(x, z, \xi) := \sum_{i=1}^{p} \big\langle \xi_i, x - y \big\rangle.$$

Theorem 2. let f_i be $c-$quasiconvex function at $x_0 \in S$ for $i \in \Delta$.

(I) If for each $y \in S$ there exists some $\xi^{(y)} \in \widehat{\partial}_c f(x_0)$ with $\varphi_y(x_0, x_0, \xi^{(y)}) \leq 0$, then $x_0 \in E$.

(II) If for each $y \in S$ there exists some $\xi^{(y)} \in \partial_c^\sharp f(x_0)$ with $\varphi_y(x_0, x_0, \xi^{(y)}) \leq 0$, then $x_0 \in W$.

Proof. **(I)** Suppose that $x_0 \notin E$. Then, we can find some $x^* \in S$ and $k \in \Delta$, satisfying

$$f_i(x^*) - f_i(x_0) \leq 0, \ \ \forall i \in \Delta, \quad \text{and} \quad f_k(x^*) - f_k(x_0) < 0. \tag{1}$$

The above inequalities and the $c-$quasiconvexy of f_i functions at x_0 imply that

$$\big\langle \xi_i, x^* - x_0 \big\rangle \leq 0, \qquad \forall i \in \Delta, \ \forall \xi_i \in \partial_c f_i(x_0). \tag{2}$$

At the other hand, the assumptions of theorem yield that there exists an $\xi^{(x^*)} \in \widehat{\partial}_c f(x_0)$ such that

$$\varphi_{x^*}(x_0, x_0, \xi^{(x^*)}) \leq 0. \tag{3}$$

It is sufficient to prove that

$$\left\langle \xi_k^{(x^*)}, x^* - x_0 \right\rangle < 0, \tag{4}$$

since (2) and (4) imply $\varphi_{x^*}(x_0, x_0, \xi^{(x^*)}) = \sum_{i=1}^{p} \left\langle \xi_i^{(x^*)}, x_0 - x^* \right\rangle > 0$, which contradicts (3).

If (4) does not hold, in view of (2) we obtain $\left\langle \xi_k^{(x^*)}, x^* - x_0 \right\rangle = 0$. By latter and $\xi_k^{(x^*)} \neq 0_n$ we can find some sequence $\{w_t\} \to x^* - x_0$ such that $\left\langle \xi_k^{(x^*)}, w_t \right\rangle > 0$ for all $t \in \mathbb{N}$. Since $w_t = (w_t + x_0) - x_0$, the latter inequality and $c-$quasiconvexity of f_k lead us to

$$\left\langle \xi_k^{(x^*)}, (w_t + x_0) - x_0 \right\rangle > 0 \implies f_k(w_t + x_0) - f_k(x_0) > 0, \quad \forall t \in \mathbb{N}.$$

Hence, the continuity of f_k concludes that:

$$\lim_{t \to \infty} \big(f_k(w_t + x_0) - f_k(x_0) \big) \geq 0 \implies f_k(x^*) - f_k(x_0) \geq 0,$$

which contradicts (1). Thus (4) holds.

(II) If $x_0 \notin W$, then there exists an $x^* \in S$ such that $f_i(x^*) - f_i(x_0) < 0$, for all $i \in \Delta$. By definition of $\partial_c^{\sharp} f(x)$, there exists a $k \in \Delta$, such that $\xi_k^{(x^*)} \neq 0_n$. Similar to the proof of (I), it can be seen that $\left\langle \xi_k^{(x^*)}, x^* - x_0 \right\rangle < 0$. The remainder of proof is similar to (I) and is hence omitted.

□

The following example shows that the converse of the above theorem does not valid.

Example 1. : Consider the following problem:

$$\begin{cases} \min \Big(|x_1| + x_1 \ , \ |x_2| + x_2 \Big) \\ \\ \quad \text{subject to} \ \ x_1 + x_2 \leq 0. \end{cases}$$

In fact, $f_1(x_1, x_2) = |x_1| + x_1$, $f_2(x_1, x_2) = |x_2| + x_2$, and $g(x_1, x_2) = x_1 + x_2$. Considering $x_0 = (0, 0)$, we have $x_0 \in E$, and

$$\partial_c f_1(x_0) = [0, 2] \times \{0\},$$
$$\partial_c f_2(x_0) = \{0\} \times [0, 2].$$

Taking $\hat{y} = (\hat{y}_1, \hat{y}_2) = (-1, -1) \in S$, for each $\xi_1^{(\hat{y})} \in \widehat{\partial}_c f_1(x_0)$ and $\xi_2^{(\hat{y})} \in \widehat{\partial}_c f_2(x_0)$, we have $\xi_1^{(\hat{y})} = (a_1, 0)$ and $\xi_2^{(\hat{y})} = (0, a_2)$ for some $a_1, a_2 \in (0, 2]$. Thus,

$$\varphi_{\hat{y}}(x_0, x_0, \xi^{(\hat{y})}) = \left\langle (a_1, 0), (-\hat{y}_1, -\hat{y}_2) \right\rangle + \left\langle (0, a_2), (-\hat{y}_1, -\hat{y}_2) \right\rangle = a_1 + a_2 > 0.$$

□

Theorem 3. If $x_0 \in E$, then for each $y \in S$ and $m \in \mathbb{N}$, there exists $z^{(m)} \in \mathbb{B}_{1/m}(x_0)$ and $\xi^{(m)} := (\xi_1^{(m)}, \ldots, \xi_p^{(m)}) \in \partial_c f(z^{(m)})$, such that

$$\left\langle \xi_i^{(m)}, y - x_0 \right\rangle \geq 0, \qquad \forall i \in \Delta, \tag{5}$$

or

$$\left\langle \xi_k^{(m)}, y - x_0 \right\rangle > 0, \qquad \exists k \in \Delta.$$

Proof. Since the proof is the same as [4, Theorem 4.2], it is omitted, An only different point of these proves is that in [4, Theorem 4.2] the feasible set is convex, and here it is not necessarily convex. □

Remark 1. The result of Theorem 3 can be written as

$$\begin{aligned} x_0 \in E \implies & \forall y \in S, \ \ \forall m \in \mathbb{N}, \ \ \exists z^{(m)} \in \mathbb{B}_{1/m}(x_0), \ \ \exists (\xi_1^{(m)}, \ldots, \xi_p^{(m)}) \in \partial_c f(z^{(m)}), \\ & \left(\langle \xi_1^{(m)}, y - x_0 \rangle, \langle \xi_2^{(m)}, y - x_0 \rangle, \ldots, \langle \xi_p^{(m)}, y - x_0 \rangle \right) \nleq 0_p. \end{aligned}$$

The similar proof of Theorem 3 shows that:

$$\begin{aligned} x_0 \in W \implies & \forall y \in S, \ \ \forall m \in \mathbb{N}, \ \ \exists z^{(m)} \in \mathbb{B}_{1/m}(x_0), \ \ \exists (\xi_1^{(m)}, \ldots, \xi_p^{(m)}) \in \partial_c f(z^{(m)}), \\ & \left(\langle \xi_1^{(m)}, y - x_0 \rangle, \langle \xi_2^{(m)}, y - x_0 \rangle, \ldots, \langle \xi_p^{(m)}, y - x_0 \rangle \right) \nless 0_p. \end{aligned}$$

Definition 3. Suppose that x_0 is an efficient solution to (P). The point $y \in S$ is said to be compatible with x_0 if the number of natural numbers m, which is satisfied in (5) is infinite. The set of all compatible points with x_0 is denoted by $S(x_0)$.

The following corollary of Theorem 3, is stated as the approximation converse of Theorem 2.

Theorem 4. Suppose that $x_0 \in E$ and $y \in S(x_0)$. Then there exists a sequence $\left\{z^{(m)}\right\}_{m=1}^{\infty}$ converging to x_0, and $\left\{\xi^{(m)}\right\}_{m=1}^{\infty}$ with $\xi^{(m)} \in \partial_c f\left(z^{(m)}\right)$, such that:

$$\varphi_y\left(x_0, z^{(m)}, \xi^{(m)}\right) \leq 0, \qquad \forall m \in \mathbb{N}.$$

Now, we introduce a new gap function for the problem (P).

Definition 4. For each $(x, z) \in S \times \mathbb{R}^n$ and $\xi := (\xi_1, \ldots, \xi_p) \in \partial_c f(z)$, the gap function $\varphi(x, z, \xi)$ is defined as:

$$\varphi(x, z, \xi) := \sup_{y \in S} \Big\{ \sum_{i=1}^{p} \langle \xi_i, x - y \rangle \Big\}.$$

It is easy to see that

$$\varphi(x, z, \xi) = \sup_{y \in S} \varphi_y(x, z, \xi).$$

Notice that the above gap function is more suitable than the gap function, which is defined in [4], because of $z = x$ in that gap function, so our gap function is its extension. Moreover, the gap function presented in [4] is more complicated in calculus, since its style is infimum of superior.

Lemma 1. For each $x \in S,\ z \in \mathbb{R}^n$, and $\xi \in \partial_c f(z)$, we have:

$$\varphi(x, z, \xi) \geq 0.$$

Proof. By taking $y = x$ in definition of $\varphi(x, z, \xi)$, the result is clear. □

Now, we can state the following famous theorem.

Theorem 5. Suppose that f_i is a $c-$quasiconvex function at $x_0 \in S$ for each $i \in \{1, \ldots, p\}$.

(I) If $\varphi(x_0, x_0, \hat{\xi}) = 0$ for some $\hat{\xi} \in \widehat{\partial}_c f(x_0)$, then $x_0 \in E$.

(II) If $\varphi(x_0, x_0, \xi^\sharp) = 0$ for some $\xi^\sharp \in \partial_c^\sharp f(x_0)$, then $x_0 \in W$.

Proof. **(I)** $\varphi(x_0, x_0, \hat{\xi}) = 0$ implies that for each $y \in S$ we have $\varphi_y(x_0, x_0, \hat{\xi}) \leq 0$. Theorem 2 justifies the result.

(II) Applying the proof of part (I), the result holds.

□

Remark 2. In the best of our knowledge, the inverse of Theorem 5 is not valid, even by convexity and differentiability of involving functions. However, in [4] shows that the inverse of Theorem 5 holds for set-valued gap function at a proper, efficient solution under some suitable assumptions. However, the characterization of situations for the satisfactory of the inverse of Theorem 5 is an important open problem.

Now, we introduce another gap function for the problem (P), in which satisfies in the converse of Theorem 5.

Definition 5. For each $x \in S$, $\xi := (\xi_1, \ldots, \xi_p) \in \partial_c f(x)$, and $\lambda := (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$, we define:

$$\varphi^*(x, \xi, \lambda) := \sup_{y \in S} \sum_{i=1}^{p} \lambda_i \langle \xi_i, x - y \rangle.$$

It is trivial that by using the proof of Theorem 5, if f_i for each $i = 1, \ldots, p$ is $c-$quasiconvex at $x_0 \in S$, and if $\varphi^*(x_0, \hat{\xi}, \lambda) = 0$ for some $\hat{\xi} \in \widehat{\partial}_c f(x_0)$ and $\lambda > 0_p$, then $x_0 \in E$. The proof of the converse of this result needs the following definition.

Definition 6. $\hat{x} \in S$ is said a Karush-Kuhn-Tucker point for problem (P) if there exist $\lambda := (\lambda_1, \ldots, \lambda_p) \geq 0_p$ with $\sum_{i=1}^p \lambda_i = 1$, and $\mu_\alpha \geq 0$ for $\alpha \in A(\hat{x})$, a finite number of them are nonzero, such that:

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_c f_i(\hat{x}) + \sum_{\alpha \in A(\hat{x})} \mu_\alpha \partial_c g_\alpha(\hat{x}).$$

$\hat{x} \in S$ is said to be strong Karush-Kuhn-Tucker point for problem (P) if the above inclusion holds for some $\lambda := (\lambda_1, \ldots, \lambda_p) > 0_p$. The set of all Karush-Kuhn-Tucker points (resp. strong Karush-Kuhn-Tucker points) of (P) is denoted by $\mathcal{K}$ (resp. $\mathcal{SK}$).

Many authors have studied necessary conditions for optimality of multiobjective semi-infinite programming; see, for example, [2, 5, 8, 9]. We can formulate these necessary conditions as follows:

$$x_0 \in W \Longrightarrow x_0 \in \mathcal{K},$$
$$x_0 \in E \Longrightarrow x_0 \in \mathcal{SK}.$$

The above mentioned necessary optimality conditions hold under some assumptions (same as closedness of *cone* $\left(\bigcup_{\alpha \in A(x_0)} \partial_c g_\alpha(x_0)\right)$ and\or compactness of index set A) and suitable constraint qualifications (same as Abadie, or Mangasarian-Fromovitz). These special conditions differ from paper to paper, and none of them play a role in proving converse of the Theorem 5, so, naturally, we use $x_0 \in \mathcal{K}$ and $x_0 \in \mathcal{SK}$ in place of $x_0 \in E$ and $x_0 \in W$.

Theorem 6. Let $x_0 \in \mathcal{K}$. If g_α functions are c-quasiconvex at x_0 for $\alpha \in A(x_0)$, then there exist $\xi \in \partial_c f(x_0)$ and $\lambda \in \mathbb{R}^n_+$ such that $\varphi^*(x_0, \xi, \lambda) = 0$.

Proof. By definition of $\mathcal{K}$, there exist some $\lambda := (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$, and nonnegative $\mu_{\alpha_1}, \ldots, \mu_{\alpha_q}$ with $\{\alpha_1, \ldots, \alpha_q\} \subseteq A(x_0)$,and $\xi_i \in \partial_c f_i(x_0)$ for $i = 1, \ldots, p$, and $\zeta_{\alpha_m} \in \partial_c g_{\alpha_m}(x_0)$ for $m = 1, \ldots, q$, such that:

$$\sum_{i=1}^{p} \lambda_i \xi_i + \sum_{m=1}^{q} \mu_{\alpha_m} \zeta_{\alpha_m} = 0. \tag{6}$$

Let $y \in S$. Then,

$$g_{\alpha_m}(y) \leq 0 = g_{\alpha_m}(x_0), \qquad \forall m = 1, \ldots, q.$$

Thus, according to c-quasiconvexity of g_{α_m} functions

$$\left\langle \zeta_{\alpha_m}, y - x_0 \right\rangle \leq 0, \qquad \forall m = 1, \ldots, q.$$

The last inequality and (6) imply that:

$$\sum_{i=1}^{p} \lambda_i \left\langle \xi_i, y - x_0 \right\rangle = -\sum_{m=1}^{q} \mu_{\alpha_m} \left\langle \zeta_{\alpha_m}, y - x_0 \right\rangle \geq 0.$$

Therefore,

$$\sum_{i=1}^{p} \lambda_i \left\langle \xi_i, x_0 - y \right\rangle \leq 0.$$

From this and $\sum_{i=1}^{p} \left\langle \xi_i, x_0 - x_0 \right\rangle = 0$, the result is proved. □

As mentioned in Remark 2, the converse of Theorem 5 is not valid in general. The following example shows this invalidity.

Example 2. Considering the problem that is considered in Example 1. we saw that $x_0 = (0, 0) \in E$ and

$$\varphi_y(x, z, \widehat{\xi}) = -a_1 y_1 - a_2 y_2,$$

for each $y = (y_1, y_2) \in S$ and $\hat{\xi}_1 = (a_1, 0)$ and $\hat{\xi}_2 = (0, a_2)$ with $a_1, a_2 \in (0, 2]$. Hence,

$$\varphi\Big(x_0, x_0, (\hat{\xi}_1, \hat{\xi}_2)\Big) = \sup\big\{ - a_1 y_1 - a_2 y_2 \mid y_1 + y_2 \leq 0 \big\}.$$

Since $a_1, a_2 > 0$, taking $y_1 < 0$ and $y_2 < 0$, implies that:

$$\varphi\Big(x_0, x_0, (\hat{\xi}_1, \hat{\xi}_2)\Big) > 0.$$

In a similar way it can be shown that for each $(\xi_1^\sharp, \xi_2^\sharp) \in \partial_c^\sharp f(x_0)$ we have

$$\varphi\Big(x_0, x_0, (\xi_1^\sharp, \xi_2^\sharp)\Big) > 0.$$

□

The following example summarizes our results.

Example 3. Consider the following problem:

$$\begin{cases} \min \left(\begin{cases} x^{\frac{1}{2}} & \text{if} \quad x \in (0,1) \\ x^{\frac{3}{2}} & \text{if} \quad x \in [1, +\infty) \\ 0 & \text{if} \quad x \in (-\infty, 0] \end{cases} , \begin{cases} x - 1 & \text{if} \quad x \in [2, +\infty), \\ 3 - x & \text{if} \quad x \in (-\infty, 2) \end{cases} \right) \\ \\ \text{subject to} \quad |x - \frac{1}{2}| - \frac{1}{2} \leq 0. \end{cases}$$

In fact, $f_1(x) = \begin{cases} x^{\frac{1}{2}} & \text{if} \quad x \in (0,1) \\ x^{\frac{3}{2}} & \text{if} \quad x \in [1,+\infty) \\ 0 & \text{if} \quad x \in (-\infty,0] \end{cases}$, $f_2(x) = \begin{cases} x-1 & \text{if} \quad x \in [2,+\infty), \\ 3-x & \text{if} \quad x \in (-\infty,2) \end{cases}$, and $g_1(x) = |x-\frac{1}{2}| - \frac{1}{2}$. It is easy to check that $\partial_c f_1(1) = \left[\frac{1}{2}, \frac{3}{2}\right]$, $\partial_c f_2(1) = \{-1\}$, $\partial_c g_1(1) = \{0\}$, and $A(1) = \{1\}$. Thus, taking $\hat{\xi} := (1,-1) \in \widehat{\partial}_c f(1)$, we conclude that $\varphi(1,1,\hat{\xi}) = 0$, and so $1 \in E$ by Theorem 5.
On the other hand, since

$$0 \in \partial_c f_1(1) + \partial_c f_2(1) + \partial_c g_1(1),$$

then $1 \in \mathcal{SK} \subseteq \mathcal{K}$ by setting $\lambda_1 = \lambda_2 = \mu_1 = 1$. This fact and Theorem 6 deduce that $\varphi^*(1,\hat{\xi},\widehat{\lambda}) = 0$ for $\widehat{\lambda} := (1,1)$. □

4 Conclusion

In conclusion, for each $x, y \in S$, $z \in \mathbb{R}^n$, $\xi_i \in \partial_c f_i(z)$, and $\lambda_i \geq 0$ with $\sum_{i=1}^{p} \lambda_i = 1$, let

$$\widehat{\varphi}_y(x,z,\xi,\lambda) := \sum_{i=1}^{p} \lambda_i \langle \xi_i, y-x \rangle,$$
$$\widehat{\varphi}(x,z,\xi,\lambda) := \sup_{y \in S} \widehat{\varphi}_y(x,z,\xi,\lambda).$$

$\widehat{\varphi}$, as a generalization of φ and φ^*, is a new general form of gap function for (P). In similar way to Theorems 3, 5, and 6 (apart from some small differences), the following theorems can be proved:

Theorem 7. Suppose that the f_i (for $i = 1, \ldots, p$) and g_α (for $\alpha \in A(x_0)$) are $c-$quasiconvex functions at x_0. Then, the following assertions hold:

(I) $\exists \hat{\xi} \in \widehat{\partial}_c f(x_0),\ \exists \lambda > 0_p,\ \widehat{\varphi}(x_0, x_0, \hat{\xi}, \lambda) = 0 \implies x_0 \in E.$

(II) $x_0 \in E \overset{suitable\ conditions}{\Longrightarrow} x_0 \in \mathcal{SK} \Longrightarrow \exists \xi \in \partial_c f(x_0),\ \exists \lambda > 0_p,\ \widehat{\varphi}(x_0, x_0, \xi, \lambda) = 0.$

Theorem 8. Suppose that the f_i (for $i = 1, \ldots, p$) and g_α (for $\alpha \in A(x_0)$) are $c-$quasiconvex functions at x_0. Then, the following assertions hold:

(I) $\exists \xi^\sharp \in \partial_c^\sharp f(x_0),\ \exists \lambda > 0_p,\ \widehat{\varphi}(x_0, x_0, \xi^\sharp, \lambda) = 0 \implies x_0 \in W.$

(II) $x_0 \in W \overset{suitable\ conditions}{\Longrightarrow} x_0 \in \mathcal{K} \Longrightarrow \exists \xi \in \partial_c f(x_0),\ \exists \lambda \geq 0_p,\ \widehat{\varphi}(x_0, x_0, \xi, \lambda) = 0.$

Theorem 9. Suppose that each f_i (for $i = 1, \ldots, p$) is a $c-$quasiconvex function at x_0. Then, the following assertions hold:

(I) $\forall y \in S,\ \exists \xi_{(y)} \in \partial_c f(x_0),\ \exists \lambda > 0_p,\ \widehat{\varphi}_y(x_0, x_0, \xi_{(y)}, \lambda) \leq 0 \implies x_0 \in E.$

(II) $x_0 \in E \Longrightarrow \forall y \in S(x_0),\ \exists \{z^{(m)}\} \to x_0,\ \exists \xi^{(m)} \in \partial_c f(z^{(m)}),\ \forall \lambda \geq 0_p,\ \widehat{\varphi}_y(x_0, z^{(m)}, \xi^{(m)}, \lambda) \leq 0 \ \ \forall m \in \mathbb{N}.$

Remark 3. It is easy to show that the condition $\exists \lambda > 0_p$ in Theorem 8(I) can be replaced by the weaker condition $\exists \lambda \geq 0_p$, if

$$\xi_k^\sharp \neq 0_n \implies \lambda_k \neq 0.$$

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Control and Optimization in Applied Mathematics (COAM)
DOI. 10.30473/coam.2019.45495.1110
Vol. 3, No. 2, Autumn-Winter 2018(13-26), ©2016 Payame Noor University, Iran

A General Scalar-Valued Gap Function for Nonsmooth Multiobjective Semi-Infinite Programming

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Received: April 14, 2019; **Accepted:** September 22, 2019.

Abstract. For a nonsmooth multiobjective mathematical programming problem governed by infinitely many constraints, we define a new gap function that generalizes the definitions of this concept in other articles. Then, we characterize the efficient, weakly efficient, and properly efficient solutions of the problem utilizing this new gap function. Our results are based on $(\Phi, \rho)-$invexity, defined by Clarke subdifferential.

Keywords. Semi-infinite programming, Multiobjective optimization, Constraint qualification, Optimality conditions, Gap function

MSC. 90C34; 90C40; 49J52.

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1 Introduction

In this paper, we consider the following multiobjective semi-infinite programming problem (MSIP):

$$\begin{array}{lll} (\mathrm{P}) & \inf \ \big(f_1(x), f_2(x), \ldots, f_p(x)\big) & \\ \text{s.t.} & g_t(x) \le 0 \quad t \in T, & x \in \mathbb{R}^n, \end{array}$$

where $f_i,\ i \in I := \{1, 2, \ldots, p\}$ and $g_t,\ t \in T$ are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$, and the index set $T \neq \emptyset$ is arbitrary, not necessarily finite. When T is finite, (P) is a multiobjective optimization problem, and when $p = 1$ and T is infinite, (P) is a semi-infinite optimization problem.

Necessary and sufficient optimality conditions for efficient, weakly efficient, and isolated efficient solutions of MSIP have been studied by many authors; see for instance [13, 18] in linear case, [12, 14] in convex case, [5] in smooth case, and [7, 11, 19, 20, 21, 23] in locally Lipshitz case. In almost all of the mentioned articles, the Karush-Kuhn-Tucker (KKT) type necessary conditions are justified for MSIPs under some constraint qualifications, and sufficient conditions are proved under several kinds of generalized convexity and generalized invexity. We know that the most general generalization of concept of invexity is $(\Phi, \rho)-$invexity, has been introduced by Caristi *et al.* in [5, 6] for smooth functions. Antczak and his coauthor presented the concept of $(\Phi, \rho)-$invexity for nonsmooth functions [1, 2], and Kanzi [19] extended this definition to a wider range of nonsmooth functions. In the present paper, we will use this most general form of $(\Phi, \rho)-$invexity.

On the other hand, the gap function for mathematical programming problems has been studied in various publications in recent years. Hearn [17] introduced a gap function for scalar convex optimization problems. Chen *et al.* [9] investigated a gap function for differentiable multiobjective optimization problems. The weak point of the gap function introduced in [9] is set-valued, i.e., brings a set to any point. Recently, Caristi *et al.* [4] can present some scalar-valued gap functions to nonsmooth multiobjective problems. Given the complexity of set-valued maps, these new single-valued gap functions are very useful. The defect gap functions introduced in [4] is that they work only for problems with convex\quasiconvex data. In the present article, this weakness will be resolved. For this end, we will define a gap function for nonsmooth MSIP, using $(\Phi, \rho)-$invexity. Of course, it should be mentioned that, in this study, if we replace "$(\Phi, \rho)-$invex" by "invex", the results will still be original which are the extensions of the existing theorems in mentioned articles.

We organize the paper as follows. In the next section, we provide the preliminary results to be used in the rest of the paper. In Section 3, we first overview some necessary optimality conditions for weakly efficient and efficient solutions, that are presented in literatures. Then, we state a similar result for properly efficien solutions. In Section 4, we introduce a new gap function involving $(\Phi, \rho)-$invexity, and present some characterizations for efficient, weakly efficient and properly efficient solutions of MSIP respect to considered gap function, unlike of other papers that consider separate gap functions for each kind of efficiency.

2 Preliminaries

In this section, we briefly overview some notions of nonsmooth analysis widely used in formulations and proofs of main results of the paper. For more details, discussion, and applications see [8].

As usual, $\langle x, y\rangle$ stands for the standard inner product $x, y \in \mathbb{R}^n$. Given $x, y \in \mathbb{R}^n$, we write $x \leq y$ (resp. $x < y$) when $x \neq y$ and $x_i \leq y_i$ (resp. $x_i < y_i$) for all $i \in \{1, \dots, n\}$. The zero vector of $\mathbb{R}^n$ is denoted by 0_n.

Given a nonempty set $A \subseteq \mathbb{R}^n$, we denote by A^0 and A^-, the polar and strictly polar cones of A, defined respectively by

$$A^0 := \{x \in \mathbb{R}^n \mid \langle x, a\rangle \leq 0, \quad \forall a \in A\},$$
$$A^- := \{x \in \mathbb{R}^n \mid \langle x, a\rangle < 0, \quad \forall a \in A\}.$$

Also, we denote the cotingent tangent cone of A at $\hat{x} \in A$ by $T(A, \hat{x})$, i.e.,

$$T(A, \hat{x}) := \big\{v \in \mathbb{R}^n \mid \exists t_r \downarrow 0,\ \exists v_r \to v \text{ such that } \hat{x} + t_r v_r \in A \ \ \forall r \in \mathbb{N}\big\}.$$

Let $\hat{x} \in \mathbb{R}^n$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of φ at $\hat{x}$ in the direction $v \in \mathbb{R}^n$, and the Clarke subdifferential of φ at $\hat{x}$ are respectively given by

$$\varphi^0(\hat{x}; v) := \limsup_{y \to \hat{x},\ t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}$$

and

$$\partial_c \varphi(\hat{x}) := \big\{\xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \leq \varphi^0(\hat{x}; v) \qquad \text{for all } v \in \mathbb{R}^n\big\}.$$

The Clarke subdifferential is a natural generalization of the classical derivative since it is known that when function φ is continuously differentiable at $\hat{x}$, $\partial_c \varphi(\hat{x}) = \{\nabla \varphi(\hat{x})\}$. Moreover when a function φ is convex, the Clarke subdifferential coincides with $\partial \varphi(\hat{x})$, the subdifferential in the sense of convex analysis, i.e.

$$\partial\varphi(\hat{x}) := \big\{\xi \in \mathbb{R}^n \mid \varphi(x) \geq \varphi(\hat{x}) + \big\langle \xi, x - \hat{x} \big\rangle \qquad \forall\, x \in \mathbb{R}^n \big\}.$$

It is worth to observe that $\partial_c\varphi(\hat{x})$ is a nonempty, convex, and compact subset of $\mathbb{R}^n$.

Theorem 1. Let ϑ_1 and ϑ_2 be locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $\hat{x} \in \mathbb{R}^n$. Then,

$$\partial_c(\alpha\vartheta_1 + \beta\vartheta_2)(\hat{x}) \subseteq \alpha\partial_c\vartheta_1(\hat{x}) + \beta\partial_c\vartheta_2(\hat{x}), \qquad \forall \alpha, \beta \in \mathbb{R}.$$

3 KKT Type Necessary Conditions

At starting point of this section, we observe that the feasible set of (P) is denoted by M, i.e.,

$$M := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0, \quad \forall t \in T\}.$$

For each $\hat{x} \in M$, set

$$F_{\hat{x}} := \bigcup_{i \in I} \partial_c f_i(\hat{x}), \quad \text{and} \quad G_{\hat{x}} := \bigcup_{t \in T(\hat{x})} \partial_c g_t(\hat{x}),$$

where, $T(\hat{x})$ denotes the set of active constraints at $\hat{x}$,

$$T(\hat{x}) := \{t \in T \mid g_t(\hat{x}) = 0\}.$$

There exist different kind of optimality, named efficiency, in multiobjective optimization. A feasible point $\hat{x}$ is said to be efficient solution [resp. weakly efficient solution] for (P) if and only if there is no $x \in M$ satisfying $f(x) \leq f(\hat{x})$ [resp. $f(x) < f(\hat{x})$]. As well as in the classical case, the KKT type optimality conditions hold at efficient and weakly efficient solutions of (P), provided some constraint qualifications (CQ) are satisfied. For example, Kanzi [20] emphasized on weakly efficiency, and introduced the CCQ as,

Definition 1. Let $\hat{x} \in S$. We say that (P) satisfies the Cottle constraint qualification (CCQ, in brief) at $\hat{x}$, if J is a compact subset of $\mathbb{R}^p$, and the function $(x, t) \to g_t(x)$ is upper semicontinuous on $\mathbb{R}^n \times T$, and $\partial^c g_t(x)$ is an upper semicontinuous mapping in t for each x, and $(G_{\hat{x}})^- \neq \emptyset$.

Then, following KKT type theorem is proved in [20, Theorem 3.6].

Theorem 2. *(KKT Necessary Condition)* Let $\hat{x} \in M$ be a weakly efficient solution of (P) and CCQ holds at $\hat{x}$. Then there exist $\alpha_i \geq 0$ (for $i \in I$) with $\sum_{i=1}^m \alpha_i = 1$, and $\beta_t \geq 0$ (for $t \in T(\hat{x})$) with $\beta_t \neq 0$ for at most finitely many indices, such that

$$0 \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \beta_t \partial_c g_t(\hat{x}).$$

Caristi and Kanzi [7] considered the efficient solutions of (P), considered a Meda type CQ as,

$$\text{(MCQ):} \qquad (F_{\hat{x}})^0 \cap (G_{\hat{x}})^0 \subseteq \bigcap_{i=1}^{p} T(Q^i, \hat{x}),$$

where, $Q^i(\hat{x}) := \Big\{x \in M \mid f_l(x) \leq f_l(\hat{x}) \quad \forall l \in I \setminus \{i\}\Big\}$, and in [7, Theorem 3.3] proved the strong KKT type result as follows.

Theorem 3. *(Strong KKT Necessary Condition).* Let $\hat{x}$ be an efficient solution of (P). If in addition, (MCQ) and the condition

$$(F_{\hat{x}})^0 \setminus \{0_n\} \subseteq \bigcup_{i=i}^{p} \big(\partial_c f_i(\hat{x})\big)^-, \tag{1}$$

hold at $\hat{x}$, then there exist scalars $\alpha_i > 0$, $i \in I$, and an integer $k \geq 0$, and a set $\{t_1, t_2, \ldots, t_k\} \subseteq T(\hat{x})$, and scalars $\beta_{t_r} \geq 0$ for $r \in \{1, 2, \ldots, k\}$, such that

$$0 \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\hat{x}) + \sum_{r=1}^{k} \beta_{t_r} \partial_c g_{t_r}(\hat{x}).$$

Also, Kanzi in [19, Theorem 3] (resp. [19, Theorem 4]) presented the KKT (resp. strong KKT) condition under Zangwill (resp. strong Zangmill) CQ, that introduced there.

Everywhere in the above, we consider the efficiency and weakly efficiency for (P). Proper efficiency is a very important notion used in studying multiobjective optimization problems. There are many definitions of proper efficiency in literature, as those introduced by Geoffrion, Benson, Borwein, and Henig; see [16] for a comparison among the main definitions of this notion. We recall the following definition from [15, pp. 110].

Definition 2. A point $\hat{x} \in M$ is called a properly efficient solution of (P) when there exists a $\lambda > 0_p$ such that

$$\langle \lambda, f(\hat{x}) \rangle \leq \langle \lambda, f(x) \rangle, \qquad \forall x \in M.$$

As proved in [10, Section 3], the above definition of proper efficiency is weaker than its other definitions (under some assumed conditions). The following theorem gives us a strong KKT condition for properly efficient solutions of (P).

Theorem 4. (Strong KKT Necessary Condition) Let $\hat{x}$ be a properly efficient solution of (P), and CCQ holds at $\hat{x}$. Then, there exist $\alpha_i > 0$ (for $i \in I$) with $\sum_{i=1}^{p} \alpha_i = 1$, and $\beta_t \geq 0$, (for $t \in T(\hat{x})$), with $\beta_t \neq 0$ for finitely many indexes, such that

$$0 \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \beta_t \partial_c g_t(\hat{x}).$$

Proof. By the definition of proper efficiency, there exist some scalars $\lambda_i > 0$ (for $i \in I$) such that

$$\sum_{i=1}^{p} \lambda_i f_i(\hat{x}) \leq \sum_{i=1}^{p} \lambda_i f_i(x), \qquad \forall x \in M.$$

This means that $\hat{x}$ is a minimizer of the following scalar semi-infinite problem:

$$\min_{x \in M} \sum_{i=1}^{p} \lambda_i f_i(x).$$

Applying Theorem 2, we get

$$0_n \in \tau \partial_c \Big(\sum_{i=1}^{p} \lambda_i f_i(\cdot) \Big)(\hat{x}) + \sum_{t \in T(\hat{x})} \mu_t \partial_c g_t(\hat{x}), \tag{2}$$

for some $\tau > 0$ and $\mu_t \geq 0$, $(t \in T(\hat{x}))$, with $\mu_t \neq 0$ for finitely many indexes. Since Theorem 1 guaranties that

$$\partial_c \Big(\sum_{i=1}^{p} \lambda_i f_i(\cdot) \Big)(\hat{x}) \subseteq \sum_{i=1}^{p} \lambda_i \partial_c f_i(\hat{x}),$$

(2) concludes that

$$0_n \in \tau \sum_{i=1}^{p} \lambda_i \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \mu_t \partial_c g_t(\hat{x}).$$

Dividing both sides of above inclusion to $\tau \sum_{i=1}^{p} \lambda_i$, we conclude that

$$0_n \in \sum_{i=1}^{p} \frac{\lambda_i}{\sum_{i=1}^{p} \lambda_i} \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \frac{\mu_t}{\tau \sum_{i=1}^{p} \lambda_i} \partial_c g_t(\hat{x}). \tag{3}$$

For each $i \in I$ and $t \in T(\hat{x})$ take

$$\alpha_i := \frac{\lambda_i}{\sum_{i=1}^{p} \lambda_i}, \quad \text{and} \quad \beta_t := \frac{\mu_t}{\tau \sum_{i=1}^{p} \lambda_i}.$$

Since $\sum_{i=1}^{p} \alpha_i = 1$, (3) completes the proof. □

We illustrate the application of Theorem 4 by an example.

Example 1. Consider the following problem:

$$\begin{array}{ll} & \inf \ (x_1, x_2) \\ \text{s.t.} & (\cos t)x_1 + (\sin t)x_2 \leq 0, \qquad t \in \big[\pi, \frac{3\pi}{4}\big]. \end{array}$$

It is easy to check that

$$M = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} + \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0\}.$$

We consider the feasible point $\hat{x} = (\cos\alpha, \sin\alpha)$ for some $\alpha \in \left(\pi, \frac{3\pi}{4}\right)$.
Since $f_1(x_1, x_2) = x_1$, $f_2(x_1, x_2) = x_2$, $g_t(x_1, x_2) = (\cos t)x_1 + (\sin t)x_2$, and $T = \left[\pi, \frac{3\pi}{4}\right]$, we get

$$T(\hat{x}) = \{\alpha\}, \qquad G_{\hat{x}} = \{(\cos\alpha, \sin\alpha)\}, \qquad F_{\hat{x}} = \{(1, 0), (0, 1)\}.$$

Therefore, according to Theorem 4, we conclude $\hat{x}$ is a properly efficient solution for the problem.

4 Characterization via gap function

This section is started by a definition from [19].

Definition 3. Suppose that the functions $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and the nonempty set $X \subseteq \mathbb{R}^n$ are given. A locally Lipschitz function $\hbar : \mathbb{R}^n \to \mathbb{R}$ is said to be $(\Phi, \rho)-$invex at $x^* \in X$ with respect to X, if for each $x \in X$ one has:

$$\Phi\big(x, x^*, 0_n, r\big) \geq 0 \qquad \text{for all } r \geq 0, \tag{4}$$

$$\Phi(x, x^*, ., .) \text{ is convex on } \mathbb{R}^n \times \mathbb{R}, \tag{5}$$

$$\Phi\Big(x, x^*, \xi, \rho(x, x^*)\Big) \leq \hbar(x) - \hbar(x^*), \qquad \forall \xi \in \partial_c \hbar(x^*). \tag{6}$$

Notice that this definition is more general that [1, Definition 4] and [2, Definition 6], since there considered ρ are real number and here is a function. Everywhere in the following, we will assume X equals to feasible solution of (P), i.e., $X = M$, but for the sake of simplicity we will omit to mention X.

Since 1982, an important function respect to convex optimization problems was defined by Hearn [17]. As mentioned in introduction, all existing literatures the gap function was defined for optimization programming with convex or quasiconvex data. Now, we define the gap function for nonsmooth MSIPs with (Φ, ρ)-invex functions.

Definition 4. Suppose that the f_i functions are (Φ, ρ_i)-invex at $x \in M$. For each

$$\xi := (\xi_1, \ldots, \xi_p) \in \prod_{i=1}^{p} \partial_c f_i(x) \text{ and } \lambda := (\lambda_1, \ldots, \lambda_p) \geq 0_p \text{ with } \sum_{i=1}^{p} \lambda_i = 1,$$

the gap function of problem (P) is defined as

$$\Upsilon(x, \xi, \lambda) := \inf_{y \in M} \left\{ \sum_{i=1}^{p} \lambda_i \Phi(y, x, \xi_i, \rho_i(y, x)) \right\}.$$

It is worth mentioning that all the gap functions considered in [7, 9, 12, 17] are special cases of above gap function. At the rest of this section, we will characterize efficient, weakly efficient, and properly efficient solutions of (P) utilizing $\Upsilon(x, \xi, \lambda)$.

Theorem 5. Let the f_i function be (Φ, ρ_i)-invex at $\hat{x} \in M$ for each $i \in I$.

(a) If $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) = 0$ for some $\hat{\xi} := (\hat{\xi}_1, \ldots, \hat{\xi}_p) \in \prod_{i=1}^{p} \partial_c f_i(\hat{x})$ and $\hat{\lambda} := (\hat{\lambda}_1, \ldots, \hat{\lambda}_p) \geq 0_p$ with $\sum_{i=1}^{p} \hat{\lambda}_i = 1$, then $\hat{x}$ is a weak efficient solution for (P).

(b) If $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) = 0$ for some $\hat{\xi} := (\hat{\xi}_1, \ldots, \hat{\xi}_p) \in \prod_{i=1}^{p} \partial_c f_i(\hat{x})$ and $\hat{\lambda} := (\hat{\lambda}_1, \ldots, \hat{\lambda}_p) > 0_p$ with $\sum_{i=1}^{p} \hat{\lambda}_i = 1$, then $\hat{x}$ is an efficient solution for (P).

Proof. (a) By contradiction assume that $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) = 0$ while $\hat{x}$ is not a weak efficient solution for (P). Then, we can find a feasible point $x_0 \in M$ such that $f_i(x_0) < f_i(\hat{x})$ for all $i \in I$. Thus, the (Φ, ρ_i)-invexity of f_i functions implies that

$$\Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho_i(x_0, \hat{x})\big) \leq f_i(x_0) - f_i(\hat{x}) < 0, \qquad \forall i \in I. \tag{7}$$

On the other hand, since $\hat{\lambda} \geq 0_p$, then there exists an index $k \in I$ such that

$$\hat{\lambda}_k > 0, \quad \text{and} \quad \hat{\lambda}_i \geq 0 \quad \forall i \in I \setminus \{k\}. \tag{8}$$

Clearly, (7) and (8) imply

$$\hat{\lambda}_k \Phi\big(x_0, \hat{x}, \hat{\xi}_k, \rho_k(x_0, \hat{x})\big) < 0, \quad \text{and} \quad \hat{\lambda}_i \Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho_i(x_0, \hat{x})\big) \leq 0 \quad \forall i \in I \setminus \{k\}.$$

Hence,

$$\sum_{i=1}^{p} \hat{\lambda}_i \Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho_i(x_0, \hat{x})\big) < 0,$$

which consequences that $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) < 0$. This contradiction completes the proof. (b) If $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) = 0$ while $\hat{x}$ is not an efficient solution for (P), there exist some $x_0 \in M$ and some index $k \in I$ such that

$$f_i(x_0) \leq f_i(\hat{x}), \quad \forall i \in I, \quad \text{and} \quad f_k(x_0) < f_k(\hat{x}).$$

According to the above inequalities, the (Φ, ρ_i)-invexity of f_i functions, and the assumption of $\hat{\lambda} > 0_p$, we get

$$\sum_{i=1}^{p} \hat{\lambda}_i \Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho_i(x_0, \hat{x})\big) \leq \sum_{i=1}^{p} \hat{\lambda}_i \left(f_i(x_0) - f_i(\hat{x})\right) < 0.$$

So, $\Upsilon(\hat{x}, \hat{\xi}, \hat{\lambda}) < 0$, which contradicts the assumption. □

Since properly efficientcy is stronger than weakly efficiency and efficiency, the following sufficient condition needs some assumptions which are stronger than Theorem 4, containing equality of ρ_i functions for each $i \in I$.

Theorem 6. Suppose that for each $i \in I$, the f_i function is (Φ, ρ)-invex at $\hat{x} \in M$. If there exists a $\hat{\xi} := (\hat{\xi}_1, \ldots, \hat{\xi}_p) \in \prod_{i=1}^{p} \partial_c f_i(\hat{x})$ such that $\Upsilon(\hat{x}, \hat{\xi}, \lambda) = 0$ for all $\lambda := (\lambda_1, \ldots, \lambda_p) > 0_p$ with $\sum_{i=1}^{p} \lambda_i = 1$, then $\hat{x}$ is a proper efficient solution for (P).

Proof. If $\hat{x}$ is not a proper efficient solution for (P), we can find some $x_0 \in M$ and $\lambda^* := (\lambda_1^*, \ldots, \lambda_p^*) > 0_p$ such that

$$\sum_{i=1}^{p} \lambda_i^* f_i(x_0) < \sum_{i=1}^{p} \lambda_i^* f_i(\hat{x}).$$

Taking $\widetilde{\lambda}_i := \frac{\lambda_i^*}{\sum_{i=1}^{p} \lambda_i^*}$, we conclude that $\sum_{i=1}^{p} \widetilde{\lambda}_i = 1$, and

$$\sum_{i=1}^{p} \widetilde{\lambda}_i f_i(x_0) < \sum_{i=1}^{p} \widetilde{\lambda}_i f_i(\hat{x}). \tag{9}$$

We claim that $\sum_{i=1}^{p} \widetilde{\lambda}_i f_i$ is a $(\Phi, \rho)-$invex function at $\hat{x}$. Suppose that $\zeta \in \sum_{i=1}^{p} \widetilde{\lambda}_i \partial_c f_i(\hat{x})$ is given. It is enough to show that

$$\Phi\big(x, \hat{x}, \zeta, \rho(x, \hat{x})\big) \leq \sum_{i=1}^{p} \widetilde{\lambda}_i f_i(x) - \sum_{i=1}^{p} \widetilde{\lambda}_i f_i(\hat{x}), \qquad \forall x \in M. \tag{10}$$

For this end, we recall from Theorem 1 that $\zeta = \sum_{i=1}^{p} \widetilde{\lambda}_i \zeta_i$ for some $\zeta_i \in \partial_c f_i(\hat{x})$. The $(\Phi, \rho)-$invexity of f_i functions at $\hat{x}$ and the convexity of $\Phi(x, \hat{x}, ., .)$ imply that

$$\begin{aligned}
\Phi\big(x, \hat{x}, \zeta, \rho(x, \hat{x})\big) &= \Phi\Big(x, \hat{x}, \sum_{i=1}^{p} \widetilde{\lambda}_i \zeta_i, \sum_{i=1}^{p} \widetilde{\lambda}_i \rho(x, \hat{x})\Big) \\
&\leq \sum_{i=1}^{p} \widetilde{\lambda}_i \Phi\big(x, \hat{x}, \zeta_i, \rho(x, \hat{x})\big) \\
&\leq \sum_{i=1}^{p} \widetilde{\lambda}_i \big(f(x) - f(\hat{x})\big) = \sum_{i=1}^{p} \widetilde{\lambda}_i f_i(x) - \sum_{i=1}^{p} \widetilde{\lambda}_i f_i(\hat{x}).
\end{aligned}$$

Thus, (10) is proved. Now, (9) and the $(\Phi, \rho)-$invexity of $\sum_{i=1}^{p} \widetilde{\lambda}_i f_i$ at $\hat{x}$ conclude that

$$\sum_{i=1}^{p} \widetilde{\lambda}_i \Phi\big(x_0, \hat{x}, \hat{\xi}_i, \rho(x_0, \hat{x})\big) \leq \sum_{i=1}^{p} \widetilde{\lambda}_i f_i(x_0) - \sum_{i=1}^{p} \widetilde{\lambda}_i f_i(\hat{x}) < 0.$$

This means $\Upsilon(\hat{x}, \hat{\xi}, \widetilde{\lambda}) < 0$, which contradicts the assumption. □

The following new definition will be required in the sequel.

Definition 5. A locally Lipschitz function $\hbar : \mathbb{R}^n \to \mathbb{R}$ is said to be "symmetric (Φ, ρ)-invex" at $\tilde{x} \in \mathbb{R}^n$ if

- $\hbar$ is (Φ, ρ)-invex at $\tilde{x}$,
- $\Phi(\tilde{x}, \tilde{x}, \xi, \rho(\tilde{x}, \tilde{x})) = 0$ for all $\xi \in \partial_c \hbar(\tilde{x})$.

$\hbar(.)$ is said to be symmetric (Φ, ρ)-invex, if it is symmetric (Φ, ρ)-invex at each point in its domain.

We recall from [23] that for r-convex ($r \in \mathbb{R}_+$) functions we have $\rho(x, y) := r$ and

$$\Phi\big(x, y, \xi, \rho\big) = \langle \xi, y - x \rangle + r\|x - y\|^2.$$

So, r-convex functions are symmetric (Φ, ρ)-invex. Also, the skew invex functions, which are defined in [22], are examples for nonconvex symmetric (Φ, ρ)-invex functions. The following example shows that a symmetric (Φ, ρ)-invexity function does not need to be invex.

Example 2. Consider a function $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(x, y, u, w) := \begin{cases} -\dfrac{u}{3y^2}|x^3 - y^3| & \text{if} \quad y \neq 0, \\ \\ w|x^3| & \text{if} \quad y = 0. \end{cases}$$

Let x and y be arbitrary elements of $\mathbb{R}$. Since $\Phi(x, y, ., .)$ is a linear function and

$$\Phi(x, y, 0, r) = \begin{cases} 0 & \text{if} \quad y \neq 0, \\ r|x^3| & \text{if} \quad y = 0, \end{cases}$$

the conditions (12) and (26) hold. Take $\rho(x, y) := -1$ for all $x, y \in \mathbb{R}$, and $\hbar(x) := x^3$. Since $\hbar(.)$ is continuously differentiable on $\mathbb{R}$, then $\partial_c \hbar(y) = \{3y^2\}$. Now, owing to

$$\begin{aligned} \Phi(x, y, 3y^2, -1) &= \begin{cases} -|x^3 - y^3| & \text{if} \quad y \neq 0, \\ -|x^3| & \text{if} \quad y = 0, \end{cases} \\ &\leq x^3 - y^3 = \hbar(x) - \hbar(y), \end{aligned}$$

we understand that $\hbar(.)$ is a $(\Phi, \rho)-$invex function at each $y \in \mathbb{R}$ with respect to $\mathbb{R}$. Also, th equality of

$$\Phi(y, y, 3y^2, -1) = 0,$$

shows that $\hbar(.)$ is a symmetric $(\Phi, \rho)-$invex function at each $y \in \mathbb{R}$. Furthermore, as it follows by [3, Theorem 1], $\hbar(.)$ is not an invex function on $\mathbb{R}$.

Theorem 7. Let $\hat{x} \in M$ be a weakly efficient solution of (P) and CCQ holds at $\hat{x}$. Suppose that for each $i \in I$ the f_i function is symmetric (Φ, ρ_i)-invex at $\hat{x}$, and for each $t \in T(\hat{x})$ the g_t function is (Φ, ρ_t)-invex at $\hat{x}$, satisfying

$$\rho_r(y, \hat{x}) \geq 0, \qquad \forall r \in I \cup T(\hat{x}),\ \forall y \in M. \tag{11}$$

Then, there exist $\xi := (\xi_1, \ldots, \xi_p) \in \prod_{i=1}^{p} \partial_c f_i(\hat{x})$ and $\lambda := (\lambda_1, \ldots, \lambda_p) \geq 0_p$ with $\sum_{i=1}^{p} \lambda_i = 1$, such that $\Upsilon(\hat{x}, \xi, \lambda) = 0$.

Proof. According to Theorem 2, we can find some $\lambda_i \geq 0$ and $\xi_i \in \partial_c f_i(\hat{x})$ (for $i \in I$) with $\sum_{i=1}^{p} \lambda_i = 1$, a finite subset T^* for $T(\hat{x})$, some $\mu_t \geq 0$ and $\zeta_t \in \partial_c g_t(\hat{x})$ (for $t \in T^*$), such that

$$\sum_{i \in I} \lambda_i \xi_i + \sum_{t \in T^*} \mu_t \zeta_t = 0_n. \tag{12}$$

For each $(i, t) \in I \times T^*$ set

$$\hat{\lambda}_i := \frac{\lambda_i}{1 + \sum_{t \in T^*} \mu_t}, \quad \text{and} \quad \hat{\mu}_t := \frac{\mu_t}{1 + \sum_{t \in T^*} \mu_t}.$$

Assume that $t \in T^*$ and $y \in M$ are arbitrarily chosen. Since $T^* \subseteq T(\hat{x})$, the (Φ, ρ_t)-invexity of g_t implies that

$$g_t(y) \leq 0 = g_t(\hat{x}) \implies \Phi\big(y, \hat{x}, \zeta_t, \rho_t(y, \hat{x})\big) \leq 0, \qquad \forall y \in M.$$

So, by $\hat{\mu}_t \geq 0$ (for $t \in T^*$), we get

$$\sum_{t \in T^*} \hat{\mu}_t \Phi\big(y, \hat{x}, \zeta_t, \rho_t(y, \hat{x})\big) \leq 0, \qquad \forall y \in M. \tag{13}$$

On the other hand, Definition 3, (11) and (12) conclude that

$$\begin{aligned} 0 \ &\leq\ \Phi\Big(y, \hat{x}, 0_n, \sum_{i \in I} \hat{\lambda}_i \rho_i(y, \hat{x}) + \sum_{t \in T^*} \hat{\mu}_t \rho_t(y, \hat{x})\Big) \\ &=\ \Phi\Big(y, \hat{x}, \sum_{i \in I} \hat{\lambda}_i \xi_i + \sum_{t \in T^*} \hat{\mu}_t \zeta_t, \sum_{i \in I} \hat{\lambda}_i \rho_i(y, \hat{x}) + \sum_{t \in T^*} \hat{\mu}_t \rho_t(y, \hat{x})\Big) nonumber \quad (14) \\ &\leq\ \sum_{i \in I} \hat{\lambda}_i \Phi\big(y, \hat{x}, \xi_i, \rho_i(y, \hat{x})\big) + \sum_{t \in T^*} \hat{\mu}_t \Phi\big(y, \hat{x}, \zeta_t, \rho_t(y, \hat{x})\big), \quad (15) \end{aligned}$$

where (15) is implied by $\sum_{i \in I} \hat{\lambda}_i + \sum_{t \in T^*} \hat{\mu}_t = 1$ and convexity of $\Phi(y, \hat{x}, ., .)$. Combining the last inequality and (13), yields

$$\sum_{i \in I} \hat{\lambda}_i \Phi\big(y, \hat{x}, \xi_i, \rho_i(y, \hat{x})\big) \geq 0 \implies \sum_{i \in I} \lambda_i \Phi\big(y, \hat{x}, \xi_i, \rho_i(y, \hat{x})\big) \geq 0, \qquad \forall y \in M. \tag{16}$$

Since the symmetric (Φ, ρ_i)-invexity of f_i functions at $\hat{x}$ concludes

$$\sum_{i\in I}\lambda_i\Phi\big(\hat{x},\hat{x},\xi_i,\rho_i(\hat{x},\hat{x})\big)=0,$$

the inequality (16) deduces that

$$\Upsilon(\hat{x},\xi,\lambda)=\inf_{y\in M}\left\{\sum_{i=1}^{p}\lambda_i\Phi\big(y,\hat{x},\xi_i,\rho_i(y,\hat{x})\big)\right\}=0,$$

as requested. □

Applying Theorems 3 and 4, and repeating the proof of Theorem 7, we can state the following theorem for efficient and properly efficient solutions of (P), respectively.

Theorem 8. Assume that $\hat{x}\in M$is an efficient solution of (P), the (MCQ) is satisfied at $\hat{x}$, and (1) holds. Suppose that for each $i\in I$ the f_i function is symmetric (Φ,ρ_i)-invex at $\hat{x}$, and for each $t\in T(\hat{x})$ the g_t function is (Φ,ρ_t)-invex at $\hat{x}$, satisfying (11). Then, there exist $\xi:=(\xi_1,\dots,\xi_p)\in\prod_{i=1}^{p}\partial_c f_i(\hat{x})$ and $\lambda:=(\lambda_1,\dots,\lambda_p)>0_p$ with $\sum_{i=1}^{p}\lambda_i=1$, such that $\Upsilon(\hat{x},\xi,\lambda)=0$.

Theorem 9. Suppose that $\hat{x}$ is a properly efficient solution for (P) and CCQ holds at $\hat{x}$. Suppose that for each $i\in I$ the f_i function is symmetric (Φ,ρ_i)-invex at $\hat{x}$, and for each $t\in T(\hat{x})$ the g_t function is (Φ,ρ_t)-invex at $\hat{x}$, satisfying (11). Then, there exist $\xi:=(\xi_1,\dots,\xi_p)\in\prod_{i=1}^{p}\partial_c f_i(\hat{x})$ and $\lambda:=(\lambda_1,\dots,\lambda_p)>0_p$ with $\sum_{i=1}^{p}\lambda_i=1$, such that $\Upsilon(\hat{x},\xi,\lambda)=0$.

We note that the difference between the Theorem7 with Theorems 8 and 9 is that in the first we have $\lambda\geq 0_p$, whereas in the latter ones we have $\lambda>0_p$. Also, it is worth mentioning that the presented results generalize

5 Conclusion

In this paper, we considered the class of nonsmooth multiobjective optimization problems with arbitrary many constraints. We proved a Karush-Kuhn-Tucker type optimality condition for properly efficient solutions of the problems. We introduced a new gap function that can characterizes efficient, weakly efficient, and properly efficient solutions the problem, under (Φ,ρ_i)-invexity and symmetric (Φ,ρ_i)-invexity assumptions.

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Control and Optimization in Applied Mathematics (COAM)
DOI. 10.30473/coam.2019.46609.1117
Vol. 3, No. 2, Autumn-Winter 2018(27-47), ©2016 Payame Noor University, Iran

MQ-Radial Basis Functions Center Nodes Selection with PROMETHEE Technique

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Received: May 31, 2019; **Accepted:** December 18, 2019.

Abstract. In this paper, we decide to select the best center nodes of radial basis functions by applying the Multiple Criteria Decision Making (MCDM) techniques. Two methods based on radial basis functions to approximate the solution of partial differential equation by using collocation method are applied. The first is based on the Kansa's approach, and the second is based on the Hermite interpolation. In addition, by choosing five sets of center nodes: Uniform grid, Cartesian, Chebyshev, Legendre and Legendre-Gauss-Lobato (LGL) as alternatives and achieving the error, condition number of interpolation matrix and memory time as criteria, rating of cases with the help of PROMETHEE technique is obtained. In the end, the best center nodes and method is selected according to the rankings. This ranking shows that Hermite interpolation by using non-uniform nodes as center nodes is more suitable than Kansa's approach with each center nodes.

Keywords. Multiple Criteria Decision Making, Radial basis functions, PROMETHEE, Hermite interpolation, Optimal selecting.

MSC. 35Q84; 74S25.

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1 Introduction

Radial basis functions (RBFs) interpolation is a technique for representing a function starting with data on scattered points. This technique first appears in the literature as a method for scattered data interpolation, and interest in this method exploded after the review of Franke [1], who found it to be the most impressive of the many methods he tested. Later, Kansa [2, 3] proposed a scheme for the estimation of partial derivatives using RBFs. The main advantage of radial basis functions methods is the meshless characteristic of them. The use of radial basis functions as a meshless method for the numerical solution of partial differential equations (PDEs) is based on the Collocation method. These methods have recently received a great deal of attention from researchers [4, 5, 6, 7, 8, 9].

Recently, RBFs methods were extended to solve various ordinary and partial differential equations including the high order ordinary differential equations [10], second-order parabolic equation with nonlocal boundary conditions [11, 12], the nonlinear Fokker-Planck equation [13], optimal control problems [14], the viscous flow over nonlinearly stretching sheet with chemical reaction, heat transfer and magnetic field [15], the unsteady flow of gas in a semi-infinite porous medium [16] nonlinear differential and integral equations [17, 18, 19], Second-order hyperbolic telegraph equation [20], the solution of 2D biharmonic equations [21], the case of heat transfer equations [22] and so on [23, 24, 25].

An RBF $\psi(\|\mathbf{x}-\mathbf{x}_i\|) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ depends on the separation between a field point $\mathbf{x} \in \mathbb{R}^d$ and the data centers $\mathbf{x}_i$, for $i = 1, 2, ..., N$, and N data points. The interpolants are classed as radial due to their spherical symmetry around centers $\mathbf{x}_i$, where $\|.\|$ is the Euclidean norm. One of the most powerful interpolation method with analytic two-dimensional test function is the RBFs method based on multiquadric (MQ) basis function

$$\psi(r) = \sqrt{r^2 + c^2}\ , \tag{1}$$

suggested by R.L. Hardy [26], where $r = \|\mathbf{x}-\mathbf{x}_i\|$ and c is a free positive parameter, often referred to as the shape parameter, to be specified by the user. Madych and Nelson [27] showed that interpolation with MQ is exponentially convergent based on reproducing kernel Hilbert space. Convergence property of the MQ has been also showed by Buhman [28, 29]. Too large or too small shape parameter c in (1) make the MQ too flat or too peaked. Despite many research works presented to finding algorithms for selecting the optimum values of c [30, 31, 32, 33, 34], the optimal choice of shape parameter is an open problem which is still under intensive investigation.

The interested reader is referred to the recent books and paper by Buhmann [28, **?**] and Wendland [35] for more basic details about RBFs, compactly and globally supported and convergence rate of the radial basis functions.

Center nodes $\{\mathbf{x}_i\}_{i=1}^N$ are not necessarily structured, that is, they can have an arbitrary distribution. The arbitrary grid structure is one of the major differences between the RBFs methods and other global methods. Such a mesh-free grid structure yields high flexibility especially when the domain is irregular. Finding the Center nodes in RBF methods is too important an open problem. In this work, we aim to select the best center nodes based on convergence, condition number of interpolation matrix, time and memory with a famous MCDM method named PROMETHEE.

Today, complex decisions in various conditions are under influence of frequent and different factors and criteria which have a significant and deniable role in consequence and effects of decisions and we cannot simply and base of the common methods find response for them but we should use (hang on to) modern scientific methods. MCDM problem is a well known branch of decision theory. It has been found in real life decision situations [36, 37, 38, 39]. In general, decision-making is the study of identifying and choosing alternatives based on the values and preferences of the decision-maker. Making a decision implies that some alternatives are to be considered, and that one chooses the alternative(s) that possibly best fits with the goals, objectives, desires and values of the problem. MCDM is a powerful tool used widely for evaluation and ranking problems containing multiple, usually conflicting, criteria [40], as how it is in finding the best center nodes in RBF methods. A lot of researchers have devoted themselves to solve MCDM [41, 42, 43, 44, 45, 46, 47, 48, 49, 50].

Several approaches have been proposed for multicriteria decision and the relevant methods were developed and applied with more or less success depending on the specific problem [51]. Among numerous methods of MCDM, The Preference Ranking Organization Method for Enrichment Evaluation (PROMETHEE) is significantly suitable for ranking applications [40]. PROMETHEE brings together flexibility and simplicity for the user [52] and is quite simple in conception and application compared to other methods for multicriteria analysis [53]. The PROMETHEE method and their applications has attracted much attention from academics and practitioners [54]. It is well adapted to problems where a finite number of alternative actions are to be ranked considering several, sometimes conflicting, criteria [51]. This method is a relatively simple ranking method, which is perfectly intelligible for the decision maker and is accepted as one of the most intuitive MCDM methods [55]. It is one of the best known and most widely applied outranking method because it follows a transparent computational procedure and can be easily understood by actors and DMs [56]. The PROMETHEE method has found a vast scope of application such as logistics and transportation [57, 58], environment management [59, 60], finance [61, 62], chemistry [63], production planning [64, 65, 66], energy management [67], service [68, 69], sport [70] and supply chain management [71, 72].

The PROMETHEE model has many advantages, in comparison to other MCDM models, such as structuring the issue, the amount of data that could be processed, the possibility to quantify the qualitative values, software support and presentation of the results [73]. Hence we used PROMETHEE Technique to rank possible alternatives due to its coordination with the structure of the issue, popularity, vast usage, remarkable outcomes, being easy to use and professional software.

This paper is arranged as follows: in Section 2, we describe the properties of radial basis functions. Two approaches based on radial basis functions for approximate the solution of linear operation by using collocation method are applied. In section 3, the PROMETHEE methodology is described. we give computational results of numerical experiments with methods based on preceding sections, to support our theoretical discussion in section 4. The conclusions are discussed in the final Section.

2 Radial basis functions

2.1 Definition of radial basis functions

Let $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$ be the non-negative half-line and let $\psi : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $\psi(0) \geq 0$. A radial basis function on $\mathbb{R}^d$ is a function of the form

$$\psi(\|\mathbf{x} - \mathbf{x}_i\|) \ ,$$

where $\mathbf{x},\ \mathbf{x}_i \in \mathbb{R}^d$ and $\|.\|$ denotes the Euclidean distance between $\mathbf{x}$ and $\mathbf{x}_i$s. If one chooses N points $\{\mathbf{x}_i\}_{i=1}^N$ in $\mathbb{R}^d$ then by custom

$$s(\mathbf{x}) = \sum_{i=1}^{N} \lambda_i \psi(\|\mathbf{x} - \mathbf{x}_i\|); \quad \lambda_i \in \mathbb{R}$$

is called a radial basis function as well [74].

2.2 RBFs interpolation based on Kansa approach

We now discuss Kansa's collocation method. Assume we are given a domain $\Omega \subset \mathbb{R}^d$, and a linear operator of the form

$$L[u](\mathbf{x}, t) = H(\mathbf{x}, t) \ , \qquad \mathbf{x} \in \Omega \ , t \in [0, T), \tag{2}$$

with initial and boundary conditions

$$I[u](\mathbf{x}) = f(\mathbf{x}) \ , \quad \mathbf{x} \in \Omega \ , t = 0, \tag{3}$$

$$B[u](\mathbf{x}) = g(\mathbf{x}, t) \ , \qquad \mathbf{x} \in \partial\Omega \ , t \in [0, T). \tag{4}$$

Then we approximate u by radial basis functions as

$$u(\hat{\mathbf{x}}) = \sum_{i=1}^{N} \lambda_i \psi(\|\hat{\mathbf{x}} - \hat{\mathbf{x}}_i\|) \ , \tag{5}$$

where $\hat{\mathbf{x}} = (\mathbf{x}, t)$. The simplest possible setting is shown in expansion (5). The Collocation matrix is constructed by matching the differential equation (2) and the initial and boundary conditions (3) and (4) at the collocation nodes $\{\hat{\mathbf{x}}_j\}_{j=1}^N$ of the form

$$A = \begin{bmatrix} B[\Psi] \\ I[\Psi] \\ L[\Psi] \end{bmatrix}, \tag{6}$$

where the blocks of matrix is generated in Appendix 1.
Kansa's method is an unsymmetric RBF Collocation method based upon the MQ interpolation functions. Although the above approach has been applied successfully in several cases [6, 7, 10,

11, 22, 75], no existence of solution and convergence analysis is available in the literature and, for some cases, it has been reported that the resulting matrix was extremely ill–conditioned. The condition number of the above interpolation matrix for smooth RBFs like Gaussian or multiquadrics are extremely large.

Several techniques have been proposed to improve the conditioning of the coefficient matrix and the solution accuracy. Fasshauer [76] suggested an alternative approach to the unsymmetric scheme based on the Hermite interpolation property of the radial basis functions. The advantage of the Hermite-based approach is that the matrix resulting from the scheme is symmetric, as opposed to the completely unstructured matrix of the same size resulting from unsymmetric schemes.

2.3 RBFs interpolation based on Hermite approach

It is possible to represent the solution u of the above boundary value problem in terms of the following Hermite RBF (HRBF) interpolation:

$$\begin{aligned} u(\hat{\mathbf{x}}) = & \sum_{i=1}^{N_0} \lambda_i B^*[\psi](\|\hat{\mathbf{x}} - \hat{\mathbf{x}}_i\|) + \sum_{i=N_0+1}^{N_1} \lambda_i I^*[\psi](\|\hat{\mathbf{x}} - \hat{\mathbf{x}}_i\|) \\ & + \sum_{i=N_1+1}^{N} \lambda_i L^*[\psi](\|\hat{\mathbf{x}} - \hat{\mathbf{x}}_i\|) \ , \end{aligned}$$

where N_0 and $N_1 - N_0$ denote the number of nodes on $\partial\Omega \times [0, T)$ and $\Omega \times \{0\}$ and $N - N_1 - N_0$ the number of internal nodes. In the above expression L^*, I^* and B^* are the operators used in (2), (3) and (4), but acting on ψ viewed as a function of the second argument $\hat{\mathbf{x}}_i$ [76]. This expansion for $u(\hat{\mathbf{x}})$ leads to a collocation matrix A which is of the form

$$A = \begin{bmatrix} B\big[B^*[\Psi]\big] & B\big[I^*[\Psi]\big] & B\big[L^*[\Psi]\big] \\ I\big[B^*[\Psi]\big] & I\big[I^*[\Psi]\big] & I\big[L^*[\Psi]\big] \\ L\big[B^*[\Psi]\big] & L\big[I^*[\Psi]\big] & L\big[L^*[\Psi]\big] \end{bmatrix} , \tag{7}$$

where the blocks generated in Appendix 2.
The matrix (7) is of the same type as the scattered HRBF interpolation matrices and thus non-singular as long as is ψ chosen appropriately. A major point in favour of the HRBF approach is that the matrix resulting from the scheme is symmetric, as opposed to the completely unstructured matrix (6) of the same size. The convergence proof for HRBF interpolation was given by Wu [77] who also recently proved the convergence of this approach when solving PDEs [78]; see also [79]. A comparison analysis between unsymmetric and symmetric radial basis function collocation methods for the numerical solution of partial differential equations is described in paper by Power [80].

3 PROMETHEE Methodology

PROMETHEE is a MCDM method developed by Brans et al. [81]. It is a ranking method quite simple in conception and application compared to other methods for multi-criteria analysis [82].

Let A be a set of alternatives and $g_j(a)$ represent the value of criterion $g_j(a),\ j = 1, 2, \cdots, J$ of alternative $a \in A$. As the first step in PROMETHEE a preference function $F_j(a, b)$ is defined for each pair of actions for criterion g_j .Assuming that more is preferred to less. Where q_i and p_i are indifference and preference thresholds for ith criterion respectively.

$$F_j(a,b) = 0 \qquad \texttt{if} \;\; g_j(a) - g_j(b) \leq q_j$$

$$F_j(a,b) = 1 \qquad \texttt{if} \;\; g_j(a) - g_j(b) \geq p_j$$

$$0 < F_j(a,b) < 1 \qquad \texttt{if} \;\; q_j < g_j(a) - g_j(b) < p_j$$

Different shapes (six types) for F_j have been suggested. If a is better than b according to jth criterion, $F_j(a, b) > 0$, otherwise $F_j(a, b) = 0$. Using the weights w_j assigned to each criterion (where $\sum w_j = 1$), one can determine the aggregated preference indicator as follows:

$$\Pi(a,b) = \sum w_j f_j(a,b).$$

If the number of alternatives is more than two, overall ranking is done by aggregating the measures of pair wise comparisons. For each alternative $a \in A$, the following two outranking dominance flows can be obtained with respect to all the other alternatives $x \in A$:

$$\varphi^+(a) = \frac{1}{n-1} \sum_{x \in A} \Pi(a,x) \quad \texttt{leaving flow}.$$

The leaving flow is the sum of the values of the arcs leaving node a and therefore provide a measure of the outranking character of a. The higher $\varphi^+(a)$, is the better alternative a,

$$\varphi^-(a) = \frac{1}{n-1} \sum_{x \in A} \Pi(x,a) \quad \texttt{entering flow}.$$

The entering flow measures the outranked character. The smaller $\varphi^-(a)$, is the better alternative a [83]. For each alternative a, it is obvious that we can also determine the net flow for each criterion separately. Let us define the net flow for criterion g_j as follows:

$$\varphi_j(a) = \frac{1}{n-1} \sum_{x \in A} (F_j(a,x) - F_j(x,a)).$$

$\varphi_j(a)$ quantifies the position of alternative a according to criterion j with respect to all the other alternatives in the set A. The larger the single criterion net flow the better alternative a on criterion g_j.

According to PROMETHEE I, action a is superior to action b if the leaving flow of a is greater than the leaving flow of b and entering flow of a is smaller than the entering flow of b.

$$a \;\; \texttt{outranks} \;\; b \;\; \texttt{if}: \; \varphi^+(a) \geq \varphi^+(b) \quad \texttt{and} \quad \varphi^-(a) \leq \varphi^-(b).$$

Equality in φ^+ and φ^- indicates indifference among the two compared alternatives. Two alternatives are considered incomparable if alternative a is better than alternative b in terms of leaving flow, while the entering flows indicate the reverse [82]:

$$[\varphi^+(a) > \varphi^+(b) \ \texttt{and} \ \varphi^-(a) > \varphi^-(b)] \ \texttt{or} \ [\varphi^+(a) < \varphi^+(b) \ \texttt{and} \ \varphi^-(a) < \varphi^-(b)].$$

PROMETHEE II provides a complete ranking of the alternatives from the best to the worst one by

$$\Phi(a) = \varphi^+(a) - \varphi^-(a).$$

The implementation of PROMETHEE requires two additional types of information, namely: (1) information on the relative importance that is the weights of the criteria considered, (2) information on the decision-maker s preference function, which he/she uses when comparing the contribution of the alternatives in terms of each separate criterion [84]. This function is used to compute the degree of preference associated to the best action in case of pairwise comparisons [85]. When we compare two alternatives a and B, we must be able to express the result of these comparisons in terms of preference. Then we consider a preference function Φ [84]. There are six basics types of preference functions proposed by Brans and Vincke [86]. with the aim of enabling the selection of specific preference function, which can be listed as usual function, U-shape function, V-shape function, level function, linear function and Gaussian function.

4 Algorithm explain with examples

The proposed approach is applied in two partial differential equations. we aim to choose best centers nodes of RBFs by applying Kansa and HRBF collocation method. Finding the best nodes between the set of nodes for example: uniform, cartesian, Chebyshev for these methods is an open problem. Thus ranking or choosing the appropriate methods by using suitable center nodes is so important in RBFs approximation.
In order to learn more about using of mentioned techniques in real environment, we impediment the proposed algorithms steps with a concrete examples.
In the process of using the model, we perform the three following steps:
1st step: Determination of fundamental criteria and Alternatives.
2nd step: Rating of cases with the help of PROMETHEE technique.
3rd step: Analyzing of consequences.

4.1 Determination of fundamental criteria and Alternatives

Here, two following classical heat equation is solved by using Kansa and HRBF method with MQ function.

$$
\begin{aligned}
&u_t(\mathbf{x},t) = \nabla u(\mathbf{x},t) + f(\mathbf{x},t), && \text{in } \Omega \times J,\\
&u(\mathbf{x},0) = g(\mathbf{x}), && \mathbf{x} \in \Omega,\\
&Bu(\mathbf{x},t) = h(\mathbf{x},t), && \text{on } \partial\Omega \times J,
\end{aligned}
$$

`Example 1: the Homogeneous one-dimensional case:`

$g(x_1) = \sin(x_1), \quad 0 < x_1 < \pi \ , \quad t > 0,$

$u(0,t) = 0, \quad u(\pi,t) = 0.$

`Exact solution:` $u(x_1,t) = \sin(x_1)\ e^{-2t}$.

`Example 2: the Inhomogeneous two-dimensional case:`

$f(x_1,x_2,t) = \sin(x_1)\sin(x_2)e^{-t} - 4,$

$g(x_1,x_2) = \sin(x_1)\sin(x_2) + x_1^2 + x_2^2, \quad 0 < x_1, x_2 < \pi \ , \quad t > 0,$

$u(0,x_2,t) = x_2^2, \quad u(x_1,0,t) = x_1^2,$

$u(\pi,x_2,t) = x_2^2 + \pi^2, \quad u(x_1,\pi,t) = x_1^2 + \pi^2.$

`Exact solution:` $u(x_1,x_2,t) = \sin(x_1)\sin(x_2)e^{-t} + x_1^2 + x_2^2$.

The error is root mean square (RMS) and obtained as:

$$RMS = \sqrt{\frac{\sum_{k=1}^{M}\left(u(\mathbf{x}_k,t_k) - u_N(\mathbf{x}_k,t_k)\right)^2}{M}} \ .$$

where $u(\mathbf{x}_k,t_k)$ and $u_N(\mathbf{x}_k,t_k)$ are achieved by exact and approximate solution on $(\mathbf{x}_k,t_k)$, and M is number of test points. Also we consider shape parameter equals one for the both examples and all cases.

Tables (1) and (2) show determination of fundamental criteria and Alternatives for each two examples.

Tables (3) and (4) show grading of cases in example 1 for $N = 36,\ 100$. Table (5) shows grading of cases in example 2 for $N = 512$.

Table 1: Fundamental criteria

Label	C_1	C_2	C_3
Criteria	Error	Condition Number	Time.Memory

Table 2: Alternatives in nodes and methods

Label	A_1	A_2	A_3	A_4	A_5
Kansa nodes	Uniform Grid	Legendre	Chebyshev	LGL	Cartesian
Label	A_6	A_7	A_8	A_9	A_{10}
HRBF nodes	Uniform Grid	Legendre	Chebyshev	LGL	Cartesian

Table 3: Grading of cases in example 1 for $N = 36$.

N	C_1	C_2	C_3
Min/Max	Min	Min	Min
Preference Function	Usual	Usual	Usual
Unit	Numerical$\times 10^{+7}$	Numerical$\times 10^{-5}$	Kbs
A_1	580	370	337.04
A_2	480	300	365.75
A_3	410	170	323.13
A_4	330	105	323.10
A_5	500	100	328.99
A_6	10	2.9	373.09
A_7	5	3.6	310.40
A_8	4	2.6	328.18
A_9	5	1.5	346.19
A_{10}	7	1.2	324.72

Table 4: Grading of cases in example 1 for $N = 100$.

N	C_1	C_2	C_3
Min/Max	Min	Min	Min
Preference Function	Usual	Usual	Usual
Unit	Numerical$\times 10^{+7}$	Numerical$\times 10^{-9}$	Kbs
A_1	4.100	190	1109.12
A_2	2.700	340	1409.06
A_3	2.600	390	1249.81
A_4	0.520	120	1285.24
A_5	20.00	17000	1124.74
A_6	0.031	15.0	1457.08
A_7	0.010	1.7	2061.45
A_8	0.003	1.1	1985.16
A_9	0.004	12.3	1984.85
A_{10}	0.090	13.0	2084.07

4.2 Rating of the cases with the help of PROMETHEE technique

In our study, one of the most frequently used preference function type in the literature and the most suitable preference function type to the characteristic of our problem, the usual function (it was introduced at Section 3) is selected for the evaluation. In next step we should evaluate

Table 5: Grading of cases in example 2 for $N = 512$.

N	C_1	C_2	C_3
Min/Max	Min	Min	Min
Preference Function	Usual	Usual	Usual
Unit	Numerical$\times 10^{+7}$	Numerical$\times 10^{-9}$	Kbs
A_1	500.00	810.0	5400
A_2	37.00	230.0	5914
A_3	71.00	130.0	6101
A_4	10.00	110.0	6010
A_5	83.00	510.0	5913
A_6	3.20	9.7	6310
A_7	0.31	3.4	6897
A_8	0.48	1.2	6911
A_9	0.17	1.1	7110
A_{10}	0.87	4.7	6981

them by analyzing the cases in each criterion, and finally by correct rating of cases, choose the best case. For this purpose, he can perform steps of PROMETHEE technique to the end or for ease of calculation; he can use the relevant software like DECISION LAB.
After completing the grading table, we can easily derive the rating consequences of the cases by using of PROMETHEE technique, Also we can evaluate and analyze the consequences by using of graphical capabilities of the software DECISION LAB, like Gaia planes.
Figure 1 displays ranking of cases with the help of PROMETHEE II technique with $N = 36$ for example 1. This ranking shows that HRBF method by using Legendre points are the most suitable choices as RBF methods and center nodes. The output figure listing the outsourcers with $N = 100$ for example 1 is given in Figure 2. As seen in the figure, the best choice in the center nodes may be changed in big number of nodes, but HRBF is the more appropriate than Kansa's method yet. Figure 3 shows PROMETHEE II output for all two scenarios $N = 36$ and $N = 100$. This ranking shows that HRBF method by using Chebyshev points as center nodes is the best choice. In Figure 4, the outsourcers are listed with $N = 512$ for example 2. This ranking shows that HRBF method by using Legendre or Legendre-Gauss-Lobatto (LGL) points as center nodes are the most suitable choices. Moreover, The geometrical analysis for interactive aid (GAIA) plane which displays the relative position of the alternatives graphically, in terms of contributions to the various criteria is given in Figures 5, 6 and 7.

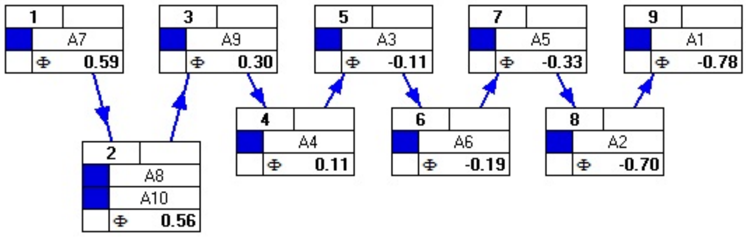

Figure 1: Example 1: Rating of cases with the help of PROMETHEE II technique with $N = 36$.

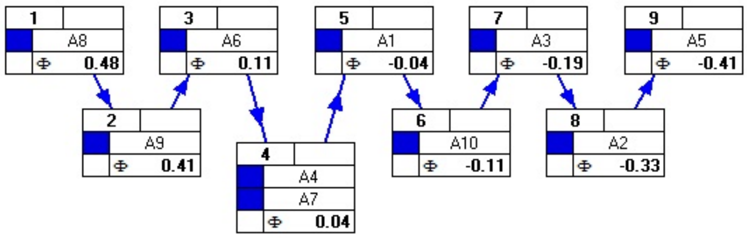

Figure 2: Example 1: Rating of cases with the help of PROMETHEE II technique with $N = 100$.

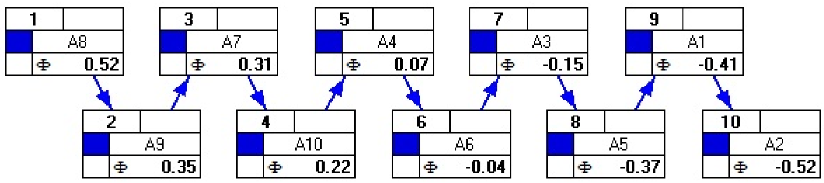

Figure 3: Example 1: PROMETHEE II output: final scores of Alternatives.

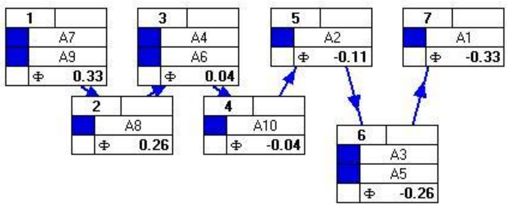

Figure 4: Example 2: Rating of cases with the help of PROMETHEE II technique with $N = 512$.

The GAIA plane was used in order to determine discriminating power of each criterion, aspects of correspondence and conflicts as well as the quality of each alternative by each criterion. Alternatives are presented by triangles and criteria by axes with square ends. Eccentric position of square of the criterion represents the volume of influence of that criterion, while correspondence between some criteria is defined by approximately the same direction of axe of those criteria. Criteria vectors expressing similar preferences on the data are oriented in the same direction, while conflicting criteria are pointing in opposite directions. The length of each

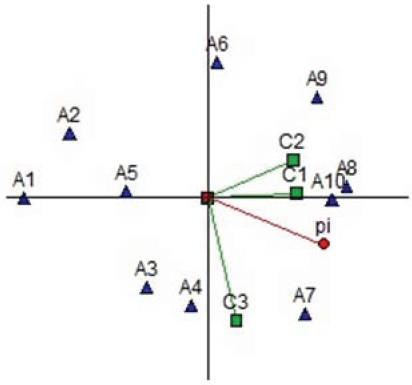

Figure 5: Example 1: Gaia planes with $N = 36$.

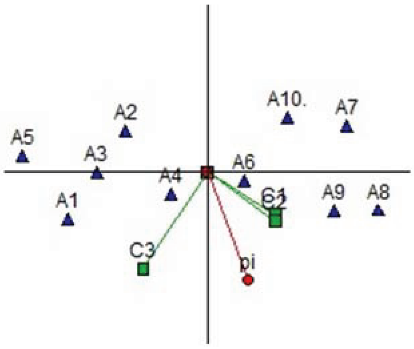

Figure 6: Example 1: Gaia planes with $N = 100$.

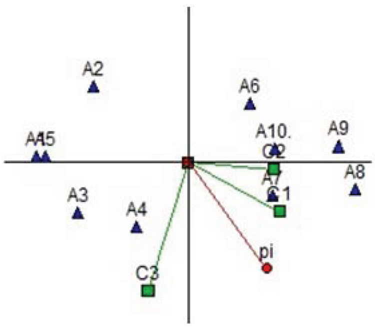

Figure 7: Example 2: Gaia planes with $N = 512$.

vector is a measure of its power in options' differentiation. Vector φ (decision axis) represents the direction of the compromise derived from the weights assignment.

4.3 consequences analysis with the help of DECISION LAB soft ware

Despite we can use potential adverse of the software in analyzing the sensitivity and determination of effectiveness of criteria validity. This capability help decision maker to observe the results of ranking when wights of criteria changed. For example, because of importance of the error in function approximations, the following figures show the consequences of rating of cases in 2 different forms with validities changed in first criteria.
Figure 8 displays of the cases according to the first weights of the criteria. Figure 9 shows of the cases according to the increase weight first criteria (0.33 to 0.50).

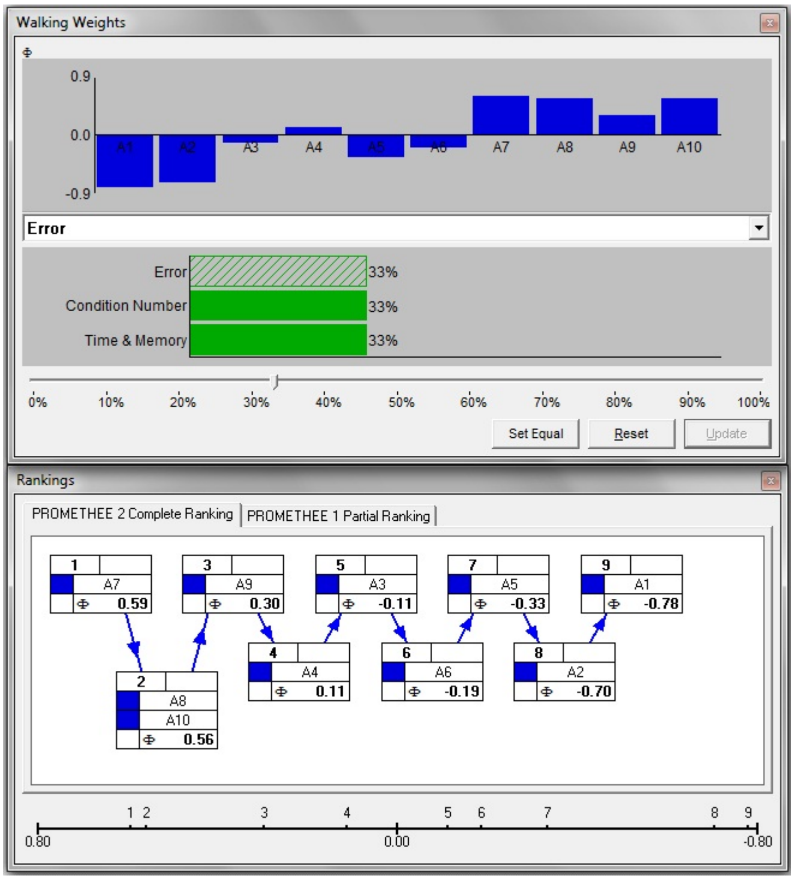

Figure 8: Example 1: Position of the cases according to the first weights of the criteria.

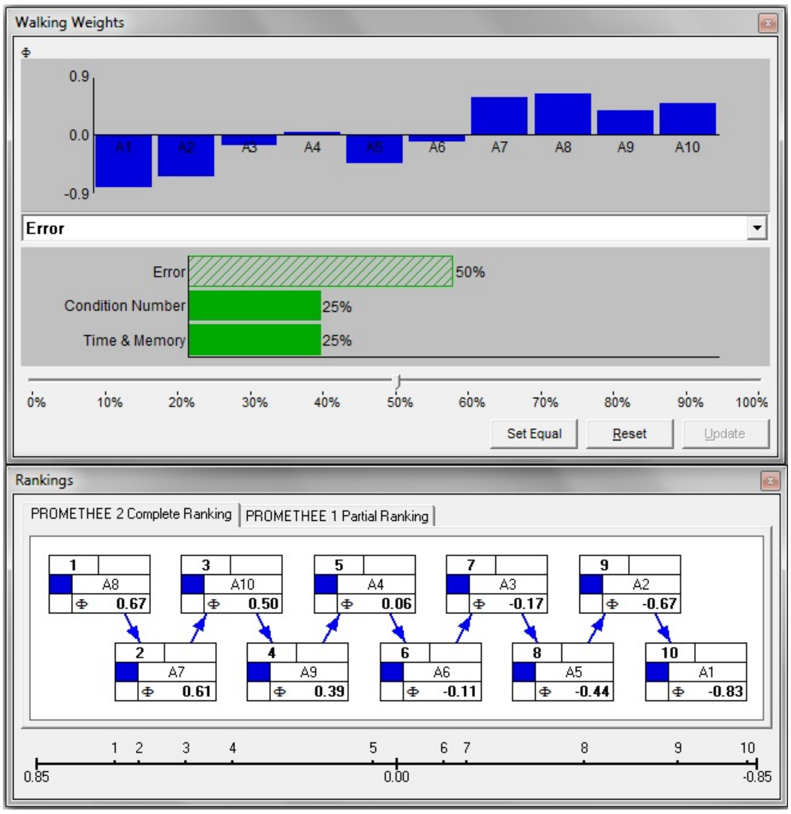

Figure 9: Example 1: Position of the cases according to the increase weight first criteria (0.33 to 0.50).

As observed, by changing the validity of the criteria, rating of the cases totally will be changed, so the applicants can evaluate the consequences of the factors validities changed in the final rating of the cases by using this method and also evaluate the effectiveness of each criterion.

5 Conclusions

Humans always are deciding in different conditions of their life and follow to find an appropriate solution for their problems; but decision making process is sometimes very complicated and necessity to assistance and counseling is unavoidable. So in the recent years, mathematical methods and knowledge of computer, as a helping decision making system has helped decision maker and create new branches and methods like MCDM techniques and decision support systems. Thus, we has used these technique in this research to optimize decision making of selecting the best radial basis functions methods and centers nodes.
Here, Two methods based on radial basis functions for approximate the solution of partial differential equation by using collocation method are applied. By choosing five sets of center

nodes: Uniform grid, Cartesian, Chebyshev, Legendre and LGL as Alternatives and achieving the error, Condition number of interpolation matrix and memory time as criteria, rating of cases with the help of PROMETHEE II technique is obtained. This ranking shows that Hermite interpolation by using non-uniform nodes as center nodes is appropriate when we applied RBF methods for solving partial differential equations.

Acknowledgements

The authors are very grateful to the referees for carefully reading the paper and for their comments and suggestions which have improved the paper. Also authors acknowledge the editor, for managing the review process for this paper.

Appendix 1.

$$\begin{aligned}
B[\Psi]_{ji} &= B[\psi](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j \in \partial\Omega \times [0,T) , \hat{\mathbf{x}}_i \in \Omega \times [0,T) , \\
I[\Psi]_{ji} &= I[\psi](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j \in \Omega \times \{0\} , \hat{\mathbf{x}}_i \in \Omega \times [0,T) , \\
L[\Psi]_{ji} &= L[\psi](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j \in \Omega^\circ \times [0,T) , \hat{\mathbf{x}}_i \in \Omega \times [0,T) .
\end{aligned}$$

Here we identify the collocation points same as center points. Ω° is interior of Ω. The problem is well-poses if the linear system $A\Lambda = C$ has unique solution [76]. C is defined of the form

$$C = \begin{bmatrix} g(\hat{\mathbf{x}}_j) \\ f(\hat{\mathbf{x}}_j) \\ H(\hat{\mathbf{x}}_j) \end{bmatrix} . \tag{8}$$

We note that a change in boundary conditions (4) is as simple as changing rows in matrix A in (6) as well as on the right hand side C in (8).

Appendix 2.

$$\begin{aligned}
B\big[B^*[\Psi]\big]_{ji} &= B\big[B^*[\psi]\big](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j, \hat{\mathbf{x}}_i \in \partial\Omega \times [0,T) , \\
B\big[I^*[\Psi]\big]_{ji} &= B\big[I^*[\psi]\big](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j \in \partial\Omega \times [0,T) , \hat{\mathbf{x}}_i \in \Omega \times \{0\}, \\
B\big[L^*[\Psi]\big]_{ji} &= B\big[L^*[\psi]\big](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j \in \partial\Omega \times [0,T) , \ \hat{\mathbf{x}}_i \in \Omega^\circ \times [0,T) , \\
I\big[B^*[\Psi]\big]_{ji} &= I\big[B^*[\psi]\big](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j \in \Omega \times \{0\} , \hat{\mathbf{x}}_i \in \partial\Omega \times [0,T), \\
I\big[I^*[\Psi]\big]_{ji} &= I\big[I^*[\psi]\big](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j, \hat{\mathbf{x}}_i \in \Omega \times \{0\}, \\
I\big[L^*[\Psi]\big]_{ji} &= I\big[L^*[\psi]\big](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j \in \Omega \times \{0\} , \ \hat{\mathbf{x}}_i \in \Omega^\circ \times [0,T) , \\
L\big[B^*[\Psi]\big]_{ji} &= L\big[B^*[\psi]\big](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j \in \Omega^\circ \times [0,T) , \hat{\mathbf{x}}_i \in \partial\Omega \times [0,T), \\
L\big[I^*[\Psi]\big]_{ji} &= L\big[I^*[\psi]\big](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j \in \Omega^\circ \times [0,T), \hat{\mathbf{x}}_i \in \Omega \times \{0\}, \\
L\big[L^*[\Psi]\big]_{ji} &= L\big[L^*[\psi]\big](\|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i\|) , && \hat{\mathbf{x}}_j, \ \hat{\mathbf{x}}_i \in \Omega^\circ \times [0,T).
\end{aligned}$$

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Control and Optimization in Applied Mathematics (COAM)
DOI. 10.30473/coam.2019.42442.1094
Vol. 3, No. 2, Autumn-Winter 2018(49-58), ©2016 Payame Noor University, Iran

Characterization of Properly Efficient Solutions for Convex Multiobjective Programming with Nondifferentiable vanishing constraints

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Received: November 18, 2018; **Accepted:** September 14, 2019.

Abstract. This paper studies the convex multiobjective optimization problem with vanishing constraints. We introduce a new constraint qualification for these problems, and then a necessary optimality condition for properly efficient solutions is presented. Finally by imposing some assumptions, we show that our necessary condition is also sufficient for proper efficiency. Our results are formulated in terms of convex subdifferential.

Keywords. Multiobjective optimization, Vanishing constraints, Convex optimization, Constraint qualification

MSC. 90C34; 90C40.

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1 Introduction

We consider the following *multiobjective mathematical programming with vanishing constraints* (MMPVC in brief):

$$\begin{aligned} \text{MMPVC}: \quad & \min_{x\in\Omega} \ F(x) := (f_1(x), \ldots, f_p(x)), \\ & \Omega := \big\{x \in \mathbb{R}^n \mid H_i(x) \geq 0, \ G_i(x)H_i(x) \leq 0, i \in I\big\}, \end{aligned} \tag{1}$$

where, the considered functions f_j (for $j \in J := \{1, \ldots, p\}$), H_i (for $i \in I := \{1, \ldots, m\}$), and G_i (for $i \in I$) are convex, not necessarily differentiable, and defined from $\mathbb{R}^n$ to $\mathbb{R}$.

If $p = 1$, then MMPVC reduces to "mathematical programming with vanishing constraints" (MPVC) which were introduced by Kanzow and his coauthors in 2007 [1, 9]. After defining the MPVC, finding the optimality conditions, named stationary conditions, for it become an interesting subject for some researchers; see [7, 8, 9, 13] in smooth case and [10, 11] in nonsmooth case).

If $G_i(x) = 0$ for $i \in I$, the MMPVC coincides to classical multiobjective programming problem which is an important field in optimization theory. Also, the MMPVC is a direct generalization for the following "mathematical problem with equilibrium constraints" (MPEC), considered in a lot of papers (see [14, 16] and their references):

$$\begin{aligned} \min \quad & F(x) & \\ s.t. \quad & H_i(x) \geq 0, \ G_i(x) \geq 0, & i \in I, \\ & G_i(x)H_i(x) = 0, & i \in I. \end{aligned}$$

To the best of our knowledge, there is no work available dealing with MMPVC with nondifferentiable data, and the present paper is the first to consider it. So far under differentiablity assumption, there is only one conference paper that considered MMPVC [12].

As well as classic multiobjective optimization, we can consider different kinds of optimality (efficiency) for MMPVC, including weakly efficient, efficient, strictly efficient, isolated efficient, and properly efficient solutions. Some characterizing of weakly efficient solutions for MMPVCs with smooth data are presented in [12]. In order to obtain optimality in which, given any objective, the trade-off between that objective and some other objective is bounded, Geoffrion [3] suggested restricting attention to efficient solutions that are proper. After Geoffrion, proper efficiency became a very important notion in studying multiobjective optimization, and many definitions for proper efficiency were introduced in literature, such as those introduced by Benson, Borwein, Henig, Kuhn-Tucker; see [2] for a comparison among the main definitions of this notion. Here, we will consider the newest definition of proper efficiency that is introduced in [4], and will characterize it for nonsmooth convex MMPVC. This characterization is made for the first time, even for MMPVCs with smooth data.

Since the product function of two convex functions is not necessarily convex, the feasible set Ω is not necessarily convex. Consequently, to set optimality conditions for properly efficient solutions of MMPVC, we can select different normal cones for S. Here we focus on Mordukhovich normal cone of Ω. This kind of optimality condition has been studied in [7, 8, 9, 14, 16] for

MPVCs and MPECs. We would mention that all mentioned references to MPVC have considered the problems with continuously differentiable functions, and the present paper extends their results to MMPVC with nondifferentiable functions.

The structure of this paper is as follows: Section 2 contains some definitions and theorems from convex analysis and non-smooth analysis. In section 3, we will introduce a new constraint qualification for MMPVC, and will present a necessary condition for properly efficient solutions of MMPVC. Then, we will show our necessary condition is also sufficient under some weak assumptions.

2 Preliminaries

In this section we present some preliminary results on convex analysis and nonsmooth analysis from [6, 15]. Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and $x_0 \in \mathbb{R}^n$. The subdifferential of g at x_0 is defined as

$$\partial g(x_0) := \{\zeta \in \mathbb{R}^n \mid g(x) - g(x_0) \geq \langle \zeta, x - x_0 \rangle, \quad \forall x \in \mathbb{R}^n\}.$$

Notice that if g_1 and g_2 are two convex functions from $\mathbb{R}^n$ to $\mathbb{R}$, and α is a non-negative real number, then $\alpha g_1 + g_2$ is convex and

$$\partial(\alpha g_1 + g_2)(x_0) = \alpha \partial g_1(x_0) + \partial g_2(x_0).$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Mordukhovich subdifferential of φ at x_0 is defined as

$$\partial_M \varphi(x_0) := \limsup_{x \to x_0} \Big\{ \xi \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{\varphi(y) - \varphi(x) - \langle \xi, y - x \rangle}{\|y - x\|} \geq 0 \Big\}.$$

We observe that if g is a convex function, then $\partial_M g(x_0) = \partial g(x_0)$ and $\partial_M(-g)(x_0) = -\partial g(x_0)$. Also, for two locally Lipschitz functions φ_1 and φ_2 from $\mathbb{R}^p$ to $\mathbb{R}$, and for an arbitrary real number α, we have

$$\partial_M \big(\alpha \varphi_1 + \varphi_2\big)(x_0) \subseteq \alpha \partial_M \varphi_1(x_0) + \partial_M \varphi_2(x_0).$$

Notice that if x_0 is a minimizer of φ on $\mathbb{R}^p$, then $0_p \in \partial_M \varphi(x_0)$, where 0_p denotes the zero vector of $\mathbb{R}^p$.

The Mordukhovich normal cone of a closed subset $\Lambda \subseteq \mathbb{R}^p$ at $x_0 \in \Lambda$ is defined by $N_M(\Lambda, x_0) := \partial_M \mathcal{I}_\Lambda(x_0)$, where

$$\mathcal{I}_\Lambda(x) := \begin{cases} 0 & x \in \Lambda, \\ +\infty & x \notin \Lambda. \end{cases}$$

It is not difficult to show that for given $\Lambda_i \subseteq \mathbb{R}^{p_i}$ and $x^{(i)} \in \Lambda_i$, $i = 1, \ldots, s$, we have

$$N_M \Big(\Lambda_1 \times \cdots \times \Lambda_s, \big(x^{(1)}, \ldots, x^{(s)}\big)\Big) = N_M\big(\Lambda_1, x^{(1)}\big) \times \cdots \times N_M\big(\Lambda_s, x^{(s)}\big). \tag{2}$$

If $h(y) = (h_1(y), \ldots, h_s(y))$, where h_is are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$, and $x^* = (x_1^*, \ldots, x_s^*)$, then the Mordukhovich coderivative of h is defined as

$$D^*h(\overline{y})(x^*) = \partial_M \left(\sum_{k=1}^{s} x_k^* h_k(y) \right) (\overline{y}).$$

Let $\Pi : \mathbb{R}^r \rightrightarrows \mathbb{R}^s$ be a set-valued function, and $\bar{x} \in \Pi(\overline{y})$. We say that Π is calm at $(\overline{y}, \overline{x})$ if there exist some $L > 0$ and neighborhoods U and V around $\overline{x}$ and $\overline{y}$, respectively, such that $d_{\Pi(\overline{y})}(x) \leq L\|y - \overline{y}\|$,for all $y \in V$ and $x \in U \cap \Pi(y)$, where $d_{\Pi(\overline{y})}(x)$ denotes the distance between x to $\Pi(\overline{y})$.

Theorem 1. [5, Theorem 4.1] Suppose that the set-valued mapping $\digamma : \mathbb{R}^l \rightrightarrows \mathbb{R}^k$ is defined as

$$\digamma(y) := \{x \in C \mid g(x) + y \in E\},$$

where the function $g : \mathbb{R}^k \to \mathbb{R}^l$ is locally Lipschitz and $(C, E) \subseteq \mathbb{R}^k \times \mathbb{R}^l$ is closed. If $\digamma$ is calm at $(0, \overline{x}) \in Gph\digamma$, then

$$N_M\big(\digamma(0), \overline{x}\big) \subseteq \bigcup_{y^* \in N_M(E, g(\overline{x}))} D^*g(\overline{x})(y^*) + N_M(C, \overline{x}).$$

Theorem 2. [5, Corollary 3.4] Consider the set-valued function $\digamma : \mathbb{R}^p \rightrightarrows \mathbb{R}^k$,

$$\digamma(y) := \{x \in \mathbb{R}^k \mid g(x, y) \in E\},$$

where $g : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^q$ is locally Lipschitz and $E \subseteq \mathbb{R}^q$ is closed. Let $(\bar{y}, \bar{x}) \in Gph\digamma$. Further, assume the following qualification condition holds,

$$\bigcup_{z^* \in N_M(E, g(\bar{x}, \bar{y})) \setminus \{0\}} [\partial_M \langle z^*, g\rangle(\overline{x}, \overline{y})]_x = \emptyset,$$

where $[\ ]_x$ denotes projection onto the x-component. Then, $\digamma$ is calm at $(\bar{y}, \bar{x})$.

For two vectors $x, y \in \mathbb{R}^p$, the inequality $x \leqq y$ stands for $x_i \leq y_i$ for all $i \in \{1, 2, \ldots, p\}$. The inequality $x \leq y$ means $x \leqq y$ and $x \neq y$. Furthermore, $x < y$ stands for $x_i < y_i$ for all $i \in \{1, 2, \ldots, p\}$.

3 Main Results

At the start of this section, we recall that the feasible solution set of MMPVC which is defined in (1) is denoted by Ω. Also, we recall the following definition from [4, pp. 110].

Definition 1. A feasible point $x_0 \in \Omega$ is called a properly efficient solution to MMPVC when there exists a vector $\lambda > 0_p$ such that

$$\langle \lambda, F(x_0)\rangle \leq \langle \lambda, F(x)\rangle, \qquad \forall x \in \Omega.$$

Throughout this paper, we fix a feasible point $\hat{x} \in \Omega$, and divide the index set I as

$$I_+ := \{i \in I \mid H_i(\hat{x}) > 0\}, \quad \text{and} \quad I_0 := \{i \in I \mid H_i(\hat{x}) = 0\}.$$

Also, we divide these two index sets as

$$I_+^0 := \{i \in I_+ \mid G_i(\hat{x}) = 0\}, \qquad I_+^- := \{i \in I_+ \mid G_i(\hat{x}) < 0\},$$
$$I_0^+ := \{i \in I_0 \mid G_i(\hat{x}) > 0\}, \qquad I_0^0 := \{i \in I_0 \mid G_i(\hat{x}) = 0\},$$
$$I_0^- := \{i \in I_0 \mid G_i(\hat{x}) < 0\}.$$

Now, we introduce a new constraint qualification for MMPVC that plays a key rule in this section.

Definition 2. The MMPVC is said to be satisfy to $(\mathfrak{CQ})$ at $\hat{x}$ if there are not, non-zero together, scalars α_i and β_i for $i \in I$, satisfying $\alpha_i \geq 0$ for $i \in I_0^0 \cup I_+^0$, $\beta_i \geq 0$ for $i \in I_0^-$, $\alpha_i\beta_i = 0$ for $i \in I_0^0$, and

$$0 \in \sum_{i \in I_0^0 \cup I_+^0} \alpha_i \partial G_i(\hat{x}) - \sum_{i \in I_0} \beta_i \partial H_i(\hat{x}).$$

We should mention that $(\mathfrak{CQ})$ is a generalization of a constraint qualification that is defined by Ye [16] for mathematical programming with equilibrium constraints (MPEC), named "No Nonzero Abnormal Multiplier Constraint Qualification". This constraint qualification was extended to nonsmooth MPECs by Movahedian and Nobakhtian [14], and is considered for MMPVC, for the first time, in the present paper.

Example 1. Let

$$\Omega = \{x \in \mathbb{R}^2 \mid x_1 \geq -x_2, \quad x_2(x_1 + x_2) \leq 0\},$$

and $\hat{x} = 0_2 \in \Omega$. This set can be considered as feasible set of a MMPVC with following data:

$$H_1(x_1, x_2) = x_1 + x_2, \qquad \text{and} \qquad G_1(x_1, x_2) = x_2.$$

Obviously, $I_0 = \{1\}$, $\partial H_1(\hat{x}) = \{(1,1)\}$ and $\partial G_1(\hat{x}) = \{(0,1)\}$. A short calculation shows that

$$0_2 \in \alpha_1 \partial G_1(\hat{x}) - \beta_1 \partial H_1(\hat{x}), \quad \alpha_1 \geq 0, \ \beta_1 \geq 0 \quad \Longrightarrow \quad \alpha_1 = \beta_1 = 0,$$

and so, the $\mathfrak{CQ}$ holds at $\hat{x}$.

The following theorem presents the first main result of this section.

Theorem 3. Let $\hat{x}$ be a properly efficient solution to MMPVC. If $(\mathfrak{CQ})$ holds at $\hat{x}$, then there exist scalars μ_j^F, μ_i^H and μ_i^G, for $j \in J$ and $i \in I$, such that:

$$0_n \in \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x}) + \sum_{i=1}^{m} \left[\mu_i^G \partial G_i(\hat{x}) - \mu_i^H \partial H_i(\hat{x})\right], \tag{3}$$

$$\mu_i^G \geq 0, \ i \in I_0^0 \cup I_+^0; \qquad \mu_i^G = 0, \ i \in I_0^+ \cup I_0^- \cup I_+^-, \tag{4}$$

$$\mu_i^H \text{ free}, \ i \in I_0^0 \cup I_0^+; \qquad \mu_i^H \geq 0, \ i \in I_0^-; \qquad \mu_i^H = 0, \ i \in I_+, \tag{5}$$

$$\mu_i^H \mu_i^G = 0, \quad i \in I_0^0, \tag{6}$$

$$(\mu_1^F, \ldots, \mu_p^F) > 0_p. \tag{7}$$

Proof. Since $\hat{x}$ is a properly efficient solution to MMPVC, Definition 1 concludes that there exist some positive scalars $\mu_j^F > 0$, for $j \in J$, such that $\hat{x}$ is a minimizer to the following weighted problem:

$$\min \sum_{j=1}^{p} \mu_j^F f_j(x) \qquad \text{subject to} \qquad x \in \Omega.$$

Therefore, $\sum_{j=1}^{p} \mu_j^F f_j + \mathcal{I}_S$ attains its global minimum at $\hat{x}$. Hence,

$$\begin{aligned} 0_n &\in \partial_M \left(\sum_{j=1}^{p} \mu_j^F f_j + \mathcal{I}_\Omega \right) (\hat{x}) \subseteq \sum_{j=1}^{p} \mu_j^F \partial_M f_j(\hat{x}) + \partial_M \mathcal{I}_\Omega(\hat{x}) \\ &= \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x}) + N_M(\Omega, \hat{x}). \end{aligned} \tag{8}$$

For estimating of $N_M(\Omega, \hat{x})$, for all $i \in I$ take $\Theta_i(x) := \big(G_i(x), H_i(x)\big)$, and let $\Theta(x) := \big(\Theta_1(x), \ldots, \Theta_m(x)\big)$. Also, set

$$\mathcal{X}_* := \{(v^1, v^2) \in \mathbb{R}^2 \mid v^2 \geq 0 \ \text{ and } v^1 v^2 \leq 0\},$$

and $\mathcal{X} := \big\{(v_1, \ldots, v_m) \in \big(\mathbb{R}^2\big)^m \mid v_i := (v_i^1, v_i^2) \in \mathcal{X}_*, \quad \forall i \in I\big\}$. Since $\mathcal{X} = \prod_{i=1}^{m} \mathcal{X}_*$, then

$$N_M\big(\mathcal{X}, \Theta(\hat{x})\big) = \prod_{i=1}^{m} N_M\big(\mathcal{X}_*, \Theta_i(\hat{x})\big), \tag{9}$$

by (2). On the other hand, the following equality has been proved in [7, Lemma 3.2]:

$$N_M(\mathcal{X}_*, \Theta_i(\hat{x})) = \begin{cases} \mathcal{X}_* & \text{for} \quad i \in I_0^0 \\ \{0\} \times \mathbb{R} & \text{for} \quad i \in I_0^+ \\ \{0\} \times \mathbb{R}_- & \text{for} \quad i \in I_0^- \\ \mathbb{R}_+ \times \{0\} & \text{for} \quad i \in I_+^0 \\ \{0\} \times \{0\} & \text{for} \quad i \in I_+^-. \end{cases} \tag{10}$$

Owing to (9)-(10), the $(\mathfrak{CQ})$ at $\hat{x}$ implies that for each $\rho = \big(\rho_1^G, \rho_1^H, \ldots, \rho_m^G, \rho_m^H\big) \in N_M\left(\mathcal{X}, \Theta(\hat{x})\right)$ we have

$$0_n \in \sum_{i \in I} \left[\rho_i^G \partial G_i(\hat{x}) + \rho_i^H \partial H_i(\hat{x})\right] \implies \rho = 0_{2m}.$$

Thus,

$$0_n \notin \bigcup_{0_{2m} \neq \rho \in N_M(\mathcal{X}, \Theta(\hat{x}))} \left[\partial \left(\langle \rho, \Theta(x) + y \rangle\right)(\hat{x}, 0_m)\right]_x.$$

From this and Theorem 2 we conclude that the set-valued function $\widehat{\Omega}(.)$ is calm at $(\hat{x}, 0_m)$, where $\widehat{\Omega}(y) := \{x \in \mathbb{R}^n \mid \Theta(x) + y \in \mathcal{X}\}$ for each $y \in \mathbb{R}^{2m}$. Since $\widehat{\Omega}(0_m) = \Omega$, Theorem 1 deduces that

$$N_M(\Omega, \hat{x}) \subseteq \bigcup_{\lambda \in N_M(\mathcal{X}, \Theta(\hat{x}))} D^*\Theta(\hat{x})(\lambda) + N_M(\mathbb{R}^n, \hat{x}). \tag{11}$$

On the other hand, by(2), for each $\lambda := \big(\lambda_1^H, \lambda_1^G, \ldots, \lambda_m^H, \lambda_m^G\big) \in \mathbb{R}^{2m}$ we have

$$D^*\Theta(\hat{x})(\lambda) = \partial_M \langle \lambda, \Theta(.)\rangle(\hat{x}) = \partial \left[\sum_{i=1}^{m} \left(\lambda_i^H H_i + \lambda_i^G G_i \right) \right] (\hat{x})$$

$$= \sum_{i=1}^{m} \left[\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x}) \right].$$

According to above equality,(11) and the fact that $N_M(\mathbb{R}^n, \hat{x}) = \{0_n\}$, we get tho following estimate for $N_M(\Omega, \hat{x})$:

$$N_M(\Omega, \hat{x}) \subseteq \bigcup_{\lambda \in N_M(\mathcal{X}, \Theta(\hat{x}))} \left[\sum_{i=1}^{m} \left(\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x}) \right) \right].$$

Hence, the last inclusion and (8) imply that

$$0_n \in \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x}) + \bigcup_{\lambda \in N_M(\mathcal{X}, \Theta(\hat{x}))} \left[\sum_{i=1}^{m} \left(\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x}) \right) \right].$$

Therefore, there exists some $\lambda := (\lambda_1^H, \lambda_1^G, \dots, \lambda_m^H, \lambda_m^G) \in N_M(\mathcal{X}, \Theta(\hat{x}))$ such that

$$0 \in \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x}) + \sum_{i=1}^{m} \left[\lambda_i^H \partial H_i(\hat{x}) + \lambda_i^G \partial G_i(\hat{x}) \right]. \tag{12}$$

From (10) and $\lambda \in N_M(\mathcal{X}, \Theta(\hat{x}))$, we can conclude that $\lambda_i^G \geq 0$ for $i \in I_0^0 \cup I_+^0$, $\lambda_i^G = 0$ for $i \in I_0^+ \cup I_0^- \cup I_+^0$, λ_i^H is free for $i \in I_0^0 \cup I_0^+$, $\lambda_i^H \leq 0$ for $i \in I_0^-$, $\lambda_i^H = 0$ for $i \in I_+^0 \cup I_+^-$, and $\lambda_i^H \lambda_i^G = 0$ for $i \in I_0^0$. Taking $\mu_i^G := \lambda_i^G$ for $i \in I$, $\mu_i^H := -\lambda_i^H$ for $i \in I_0^0$, $\mu_i^H := \lambda_i^H$ for $i \in I \setminus I_0^0$, and considering (12), the result is justified. □

It is worth mentioning that when $p = 1$, the relations (3)-(7), named M-stationary condition, are proved in [7, 8] for the problems with smooth data, and in [14] for nonsmooth MPECs. The present paper is the first that studies this kind of stationary condition for MMPVCs.

We know from classic nonlinear optimization that necessary optimality conditions are also to be sufficient under convexity assumption. These results cannot be applied for MMPVC since the product function H_iG_i does not convex. The following theorem, which is our second main result in this section, shows the sufficient condition holds for MMPVCs, under some additional weak assumptions.

Theorem 4. Let $\hat{x} \in \Omega$ be a feasible solution that satisfies in (3)-(7) for some scalars μ_j^F, μ_i^H, and μ_i^G, $(i, j) \in I \times J$.

(a): If

$$\mathcal{A} := \{i \in I_0^0 \mid \mu_i^H < 0\} \cup \{i \in I_0^0 \mid \mu_i^H = 0,\ \mu_i^G > 0\} = \emptyset,$$

then $\hat{x}$ is a local properly efficient to MMPVC.

(b): If

$$\mathcal{B} := \mathcal{A} \cup \{i \in I_0^+ \mid \mu_i^H < 0\} \cup \{i \in I_+^0 \mid \mu_i^H = 0,\ \mu_i^G > 0\} = \emptyset,$$

then $\hat{x}$ is a global properly efficient to MMPVC.

Proof. **(a):** Suppose that $\hat{x}$ is not locally properly efficient to MMPVC. Then, for each neighborhood $U \subseteq \mathbb{R}^n$ to $\hat{x}$, and for each vector $\lambda = (\lambda_1, \dots, \lambda_p) > 0_p$, we can find a point $x_\lambda^U \in \Omega \cap U$ such that

$$\sum_{j=1}^{p} \lambda_j f_j(\hat{x}) > \sum_{j=1}^{p} \lambda_j f_j(x_\lambda^U).$$

Notice that (7) leads us take $\lambda = \mu^F := (\mu_1^F, \dots, \mu_p^F)$ in above inequality. So, the convexity of $\sum_{j=1}^{p} \mu_j^F f_j$ implies that

$$\langle \varsigma, x_\mu^U - \hat{x} \rangle \leq \sum_{j=1}^{p} \mu_j^F f_j(x_\mu^U) - \sum_{j=1}^{p} \mu_j^F f_j(\hat{x}) < 0, \qquad \forall \varsigma \in \partial \left(\sum_{j=1}^{p} \mu_j^F f_j \right) (\hat{x}).$$

The last inequality and the fact that $\partial \left(\sum_{j=1}^{p} \mu_j^F f_j \right) (\hat{x}) = \sum_{j=1}^{p} \mu_j^F \partial f_j(\hat{x})$ conclude that

$$\sum_{j=1}^{p} \mu_j^F \langle \varsigma_j, x_\mu^U - \hat{x} \rangle < 0, \quad \exists x_\mu^U \in U \cap \Omega, \ \forall \varsigma_j \in \partial f_j(\hat{x}). \tag{13}$$

On the other hand, (3) implies that

$$\sum_{j=1}^{p} \mu_j^F \xi_j^F + \sum_{i=1}^{m} (\mu_i^G \xi_i^G - \mu_i^H \xi_i^H) = 0, \tag{14}$$

for some $\xi_j^F \in \partial f_j(\hat{x})$, $\xi_i^H \in \partial H_i(\hat{x})$ and $\xi_i^G \in \partial G_i(\hat{x})$, for $(i, j) \in I \times J$.

Let $i \in I_0^+$. The continuity of G_i concludes that there exists a neighborhood U_i for $\hat{x}$ such that $G_i(x) > 0$ for all $x \in U_i$. Thus, $G_i(x) > 0$, $H_i(x) \geq 0$ and $G_i(x)H_i(x) \leq 0$, for all $x \in U_i \cap \Omega$, which imply $H_i(x) = 0$. Similarly, for each $i \in I_+^0$ there exists a neighborhood $\widehat{U}_i$ for $\hat{x}$ such that $H_i(x) > 0$ and $G_i(x) \leq 0$. Summarizing, for all $x \in \Omega \cap V$ in which $V := \bigcap_{i \in I_0^+} U_i \cap \bigcap_{i \in I_0^+} \widehat{U}_i$, we have $G_i(x) \leq 0 = G_i(\hat{x})$, for $i \in I_+^0$, and $H_i(x) = 0 \leq H_i(\hat{x})$, for $i \in I_0^+$. Hence

$$\langle \xi_i^G, x - \hat{x} \rangle \leq 0, \quad \forall i \in I_+^0, \quad \text{and} \quad \langle \xi_i^H, x - \hat{x} \rangle \leq 0, \quad \forall i \in I_0^+.$$

So, owing to (4)-(6), we get

$$\langle \sum_{i \in I_+^0 \cup I_0^+} (\mu_i^G \xi_i^G - \mu_i^H \xi_i^H), x - \hat{x} \rangle \leq 0, \qquad \forall x \in \Omega \cap V.$$

By the above inequality, convexity of functions, assumption that $\mathcal{A} = \emptyset$, (4)-(6), and a short calculation, we deduce that

$$\langle \sum_{i=1}^{m} (\mu_i^G \xi_i^G - \mu_i^H \xi_i^H), x - \hat{x} \rangle \leq 0, \qquad \forall x \in \Omega \cap V. \tag{15}$$

Now, inner-producing two sides of (14) to $x - \hat{x}$ and regarding (15), we conclude that

$$\sum_{j=1}^{p} \mu_j^F \langle \xi_j^F, x - \hat{x} \rangle \geq 0, \qquad \forall x \in \Omega \cap V,$$

which contradicts (13). Thus, the proof is complete.

(b): Emptiness assumption of $\mathcal{B}$ leads us to repeat the proof of (a) without considering any neighborhood for $\hat{x}$.

□

Example 2. Consider the MMPVC with following data:

$$\begin{aligned} f_1(x_1,x_2) &= x_1^2+|x_2|, & f_2(x_1,x_2) &= 2x_1^4+3|x_2|,\\ H_1(x_1,x_2) &= -x_2, & H_2(x_1,x_2) &= |x_1|+x_2,\\ G_1(x_1,x_2) &= -1, & G_2(x_1,x_2) &= -x_1. \end{aligned}$$

Taking $\hat{x}=0_2$, we conclude that $I_0^- = \{1\}$ and $I_0^0=\{2\}$. Since the conditions (3)-(7) hold for $\mu_1^F=\mu_2^F=1$, $\mu_1^H=\mu_2^H=\frac{1}{4}$ and $\mu_1^G=\mu_2^G=0$, and also $\mathcal{B}=\emptyset$, Theorem 4 implies that $\hat{x}$ is properly sufficient for the problem.

4 Conclusion

In this paper, we considered a new class of nonsmooth multiobjective optimization problems, denoted by MMPVC, as an extension of the mathematical programs with vanishing constraints from the scalar case and the multiobjective mathematical programming with equilibrium constraints. We introduced a suitable modification of the "No Nonzero Abnormal Multiplier Constraint Qualification". We gave Karush-Kahn-Tucker type necessary optimality condition for proper efficient solutions, and derived that this necessary condition is also sufficient for proper efficiency under some additional assumptions in emptiness kind.

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Control and Optimization in Applied Mathematics (COAM)
DOI. 10.30473/coam.2020.46435.1116
Vol. 3, No. 2, Autumn-Winter 2018(59-76), ©2016 Payame Noor University, Iran

Two Settings of the Dai-Liao Parameter Based on Modified Secant Equations

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Received: May 28, 2019; **Accepted:** February 18, 2020.

Abstract. Following the setting of the Dai-Liao (DL) parameter in conjugate gradient (CG) methods, we introduce two new parameters based on the modified secant equation proposed by Li et al. (Comput. Optim. Appl. 202:523-539, 2007) with two approaches, which use an extended new conjugacy condition. The first is based on a modified descent three-term search direction, as the descent Hestenes-Stiefel CG method. The second is based on the quasi-Newton (QN) approach. Global convergence of the proposed methods for uniformly convex functions and general functions is proved. Numerical experiments are done on a set of test functions of the CUTEr collection and the results are compared with some well-known methods.

Keywords. Unconstrained optimization, Modified secant equations, Dai-Liao conjugate gradient method

MSC. 90C34; 90C40.

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1 Introduction

Conjugate gradient (CG) and quasi-Newton (QN) methods contain a class of unconstrained optimization algorithms, with some great properties such as low memory requirements and strong global convergence [34], which make them famous for engineers and mathematicians engaged in solving large-scale problems, as follows:

$$\begin{aligned} &\min f(x) \\ &x \in \mathbb{R}^n \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth nonlinear function, and its gradient is available. The iterative formula of a CG method leads to a sequence of the approximate solutions, as $\{x_n\}$ with the following recursive formula:

$$x_{k+1} = x_k + s_k, \quad s_k = \alpha_k d_k, \quad k = 0, 1, 2, \ldots \tag{2}$$

where $x_0 \in \mathbb{R}^n$ is an initial solution and d_k is the search direction with following formula:

$$d_0 = -g_0, \quad d_{k+1} = -g_{k+1} + \beta_k d_k, \quad k = 0, 1, 2, \ldots \tag{3}$$

where $g_k = \nabla f(x_k)$ and β_k is a scalar called the CG (update) parameter. In Eqn (2) the α_k parameter is the step length at current iteration along d_k. Inexact line searches satisfy some certain line search conditions [22]. Among them, the so-called Wolfe conditions [22] have attracted particular attention in the convergence analyses and the implementation of CG methods, requiring that:

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^T d_k, \tag{4}$$

$$g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k \tag{5}$$

where $0 < \delta < \sigma < 1$. These conditions guarantee that $s_k^T y_k > 0$, where $y_k = g_{k+1} - g_k$, and s_k is defined in (2).

Different choices for the CG parameters lead to different CG methods. In early CG methods, the conjugate condition is based on the quadratic objective function and the exact line search, which is $d_k^T g_{k+1} = 0$. These methods lead to the classical linear CG methods such as Fletcher-Reeves (FR) [23], Hestenes-Stiefel (HS) [21], Polak-Ribie´re-Polyak (PRP)[9, 13] and Dai-Yuan (DY)[36]. Classic methods have same performance for linear CG methods, although they have different global convergence properties and numerical performance for general nonlinear objective functions or inexact line search (see [32]).

New nonlinear CG methods are presented with different approaches such as constructing descent or sufficient descent directions, new extended conjugacy conditions or a hybrid with QN methods. For example, Zhang et al. [18], construct some descent classic CG directions as three- terms CG, TTCG, methods. For instance in a special case, they proposed a three-term HS, TTHS, with the following search direction [18]:

$$d_{k+1}^{TTHS} = -g_{k+1} + \beta_k^{HS} d_k - \theta_{k+1} y_k, \tag{6}$$

where

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k}, \quad \theta_{k+1} = \frac{g_{k+1}^T d_k}{d_k^T y_k} \tag{7}$$

It is also clear that if the exact line search is used, then $\theta_{k+1} = 0$, and the TTHS method is converted to the classic HS method. By replacing the HS method with other linear CG methods, some new descent methods, such as TTPR and TTFR can be achieved (see [18]). An attractive feature of these methods is that the direction has sufficient descent conditions, i.e. $d_k^T g_k = -\|g_k\|^2$ ($\|.\|$ is the Euclidean norm), which is independent of line search [18]. In addition, Babaie-Kafaki and Ghanbari [28] apply the idea of TTHS method, Eqns (6)-(7), using a modified BFGS, proposed by Li and Fukushima [6], and introduce a modified TTCG, named MTTHS, as follows:

$$d_{k+1}^{MTTHS} = -g_{k+1} + \beta_k^{MHS} d_k - \theta_k^M z_k, \tag{8}$$

where

$$\beta_k^{MHS} = \frac{g_{k+1}^T z_k}{d_k^T z_k}, \quad \theta_k^M = \frac{g_{k+1}^T d_k}{d_k^T z_k}, \tag{9}$$

and

$$z_k = g_{k+1} - g_k + c\|g_k\|^r s_k \triangleq y_k + c\|g_k\|^r s_k, \tag{10}$$

where $r \geq 0$ and $c > 0$ are some constants, in Eqn (10), z_k plays a vital role in the global convergence of the MBFGS method for nonconvex function [17]. Similarly, Sugiki et al. [15] proposed another modified TTCG method, using a TTCG method, proposed by Narushima et al. [37] and a general form of the modified secant conditions, which generate a search direction with sufficient descent conditions.

At first time, Perry [3] to find more efficient CG methods, incorporated the standard secant equation to conjugacy condition and proposed his method to approximate the directions of CG to QN direction, as in the following:

$$d_{k+1}^P = -g_{k+1} + \beta_k^P d_k = -Q_{k+1}^P g_{k+1}, \tag{11}$$

where Q_{k+1}^P is the direction matrix, as a nonsymmetric matrix which approximates the inverse Hessian of the objective function at current iteration, and β_k^P is the Perry CG parameter, which are defined as follows:

$$\beta_k^P = \frac{g_{k+1}^T y_k}{d_k^T y_k} - \frac{g_{k+1}^T s_k}{d_k^T y_k}, \quad Q_{k+1}^P = I - \frac{s_k y_k^T}{y_k^T s_k} + \frac{s_k s_k^T}{y_k^T s_k} \tag{12}$$

As mentioned, from Wolfe conditions in means (4)-(5), we have $s_k^T y_k > 0$, so the matrix in (12) is well-defined. In Perry approach, the direction matrix, Q_{k+1}^P, is not symmetric and also does not satisfy the secant equations [5]. To overcome these defects, Shanno [5] combined the Perry method and memoryless BFGS method to introduce a new CG direction as follows:

$$d_{k+1}^S = -Q_{k+1}^S g_{k+1} = -(I - \frac{s_k y_k^T + y_k s_k^T}{s_k^T y_k} + (1 + \frac{y_k^T y_k}{s_k^T y_k})\frac{s_k s_k^T}{s_k^T y_k})g_{k+1} \tag{13}$$

In 2001, Dai, and Liao [35] extended the Perry conjugate condition and introduced the new nonlinear CG method as follows:

$$d_{k+1}^{DL} = -g_{k+1} + \beta_k^{DL} d_k = -Q_{k+1}^{DL} g_{k+1}, \tag{14}$$

where

$$\beta_k^{DL} = \frac{g_{k+1}^T y_k}{d_k^T y_k} - t\frac{g_{k+1}^T s_k}{d_k^T y_k}, \quad Q_{k+1}^{DL} = I + \frac{s_k^T y_k}{s_k^T y_k} - t\frac{s_k^T s_k}{s_k^T y_k}, \tag{15}$$

where t is a nonnegative DL parameter. Note that if t =0, then β_k^{DL} reduces β_k^{HS}, Eqn(7), if $t = 1$, then β_k^{DL} reduces to β_k^{P}, Eqn(12).

For extending the global convergence properties of general objective functions, Dai, and Liao [35] considered a truncated form of the DL method, with an extended DL parameter, namely β_k^{DL+}, and the following direction:

$$d_{k+1}^{DL+} = -g_{k+1} + \beta_k^{DL+} d_k = -g_{k+1} + \left(\max\{\frac{g_{k+1}^T y_k}{d_k^T y_k}, 0\} - t\frac{g_{k+1}^T s_k}{d_k^T s_k}\right) d_k \tag{16}$$

As a famous descent CG method, independent from a type of line search, Hager and Zhang (HZ) [31] introduced the following CG parameter:

$$\beta_k^{HZ} = \frac{g_{k+1}^T y_k}{d_k^T y_k} - 2\frac{\|y_k\|^2}{d_k^T y_k}\frac{g_{k+1}^T d_k}{d_k^T y_k} \tag{17}$$

HZ method is an adaptive version of the DL parameter corresponding to $t = 2\frac{\|y_k\|^2}{s_k^T y_k}$ in Eqn (15). Another adaptive DL parameter is based on scaled memoryless BFGS, suggested by Dai and Kou (DK) [33], as follows:

$$\beta_k^{DK}(\tau_k) = \frac{g_{k+1}^T y_k}{d_k^T y_k} - (\tau_k + \frac{\|y_k\|^2}{s_k^T y_k} - \frac{s_k^T y_k}{\|s_k\|^2})\frac{g_{k+1}^T s_k}{d_k^T y_k} \tag{18}$$

In which τ_k is a parameter corresponding to the scaling factor in the scaled memoryless BFGS method.

Although the setting of the DL parameter is an open problem in CG methods [2], many efforts have been made by researchers to adjust it. As instance, in descent approach based on an eigenvalue study, the authors in [25] proposed a descent class of DL method, namely, DDL. An exciting feature of the proposed class is that the HZ and DK methods are individual cases of it, as efficient nonlinear CG methods. The DDL search direction is as follows [25]:

$$d_{k+1}^{DDL} = -g_{k+1} + \beta_k^{p,q} d_k = -(I + \frac{d_k^T y_k}{d_k^T y_k} - t_k^{p,q}\frac{d_k^T s_k}{d_k^T y_k}) g_{k+1}, \tag{19}$$

where $t_k^{p,q}$ is DL parameter as follows:

$$t_k^{p,q} = p\frac{\|y_k\|^2}{s_k^T y_k} - q\frac{s_k^T y_k}{\|s_k\|^2}, \tag{20}$$

where p and q are nonnegative constants, which $p < \frac{1}{4}$ and $q \geq \frac{1}{4}$. For more information about setting the DL parameter, see [16, 24, 27, 29, 38].

Another conjugacy approach in CG methods is based on the different types of modified secant equations instead of standard secant equation in DL method. To review the different types of modified secant equations see the Introduction section of [25].

Here, motivated by $DL+$ approach, similar to [17, 13, 10, 11], we apply the modified secant equation proposed by Li et al. [10], named MSL, for a new extended conjugacy condition and then using two approaches, similar to $DL+$, we adjust its parameter. Therefore, the advantages of the new proposed nonlinear CG method are using the second-order information of the objective function, by a modified secant equation, and setting the $DL+$ parameter to improve in the search directions, simultaneously.

The remainder of this paper is organized as follows. In Section 2, we introduce a new extended conjugacy condition based on MSL [10]. Then we discuss two approaches to setting the parameter. In the first approach, we use the MTTHS descent method (8)-(9). In second approach, we try to match the direction matrix of the CG method to the Shanno quasi-Newton direction matrix, Q^S_{k+1}, Eqn (13). Then, we discuss their global convergence. In Section 3, we numerically compare our methods with the DL, HZ, and DK methods and report comparative testing results. Finally, we make conclusions in Section 4.

2 New Nonlinear Conjugate Gradient Methods

In this section, based on MSL [10], we first introduce a new extended, modified conjugate condition for CG methods, and then we describe two methods for calculating the parameter.

2.1 Conjugacy condition based on MSL

Using modifies secant equations are common in CG and QN methods for solving unconstrained optimization problems. For example Zhang et al. [13] and Zhang and Xu [14] proposed new QN methods based on a modified secant equation. Moreover, Yube, and Takano [11] applied this equation for a nonlinear CG with global convergence properties. New versions of this modified secant equation can be seen in [26, 20, 12]. Zhang and Zhou [17] applied a modified BFGS method for a nonlinear CG method, which is proposed by Li and Fukushima [6]. Li et al. [10] used with the modified secant equation in [39, 40]. Suugiki et al. [15], unify the above-modified secant equations as a general form and proposed a TTCG method with sufficient descent property.

As special case, here, we apply the conjugacy condition proposed by Li et al. [10], which further studied by [39, 40]. This condition is based on the modified secant equation, MSL, as follows [10]:

$$B_{k+1}s_k = \overline{y}_k, \quad \overline{y}_k = y_k + A_k u_k, \tag{21}$$

where B_{k+1} is an approximation of the Hessian matrix of the objective function, $u_k \in \mathbb{R}^n$ is a vector that satisfies $s_k^T u_k \neq 0$ and $A_k = \frac{\bar{\theta_k}}{s_k^T u_k}$ where $\bar{\theta_k} = max\{\theta_k, 0\}$ and θ_k is as follows:

$$\theta_k = 2(f_k - f_{k+1}) + (g_k + g_{k+1})^T s_k \tag{22}$$

The modified secant equation in Eqn (21), is based on a revised form of the modified secant equation proposed in [39, 40]. According to (3), similar to DL conjugate condition [35], the new extended conjugacy condition based on (21) is presented as follows:

$$d_{k+1}^T \overline{y}_k = -t^{\overline{DL+}} g_{k+1}^T s_k, \tag{23}$$

which is named $\overline{DL+}$ conjugate condition. Using CG direction in (3) and (23), we have the following CG parameter:

$$\beta_k^{\overline{DL+}} = \frac{g_{k+1}^T \overline{y}_k}{d_k^T \overline{y}_k} - t^{\overline{DL+}} \frac{g_{k+1}^T s_k}{d_k^T \overline{y}_k} \tag{24}$$

For $t^{\overline{DL+}} = 0$, the $\overline{DL+}$ method is converted to the MHS method in Eqn (9). By replacing the (24) in (3) and rearranging the vectors, we have the following new search direction:

$$d_{k+1}^{\overline{DL+}} = -Q_{k+1}^{\overline{DL+}} g_{k+1} = -(I + \frac{s_k^T \overline{y}_k}{s_k^T \overline{y}_k} - t^{\overline{DL+}} \frac{s_k^T s_k}{s_k^T \overline{y}_k}) g_{k+1} \tag{25}$$

Then the associate CG method is called $\overline{DL+}$ and its parameter, $t^{\overline{DL+}}$, is called $\overline{DL+}$ (update) parameter.
Now similar to $DL+$ parameter, the setting of the $\overline{DL+}$ parameter is an vital issue. In following, we use two approaches to set it.

2.2 Setting $\overline{DL+}$ parameter

To set the $\overline{DL+}$ parameter, we apply two approaches. The first is based on the descent direction, and the second is based on the QN approach.

2.2.1 Descent approach

In linear search methods, the descent direction is vital to convergence analysis. Since the $\overline{DL+}$ direction may not satisfy the descent condition, similar to [25] for DL method, here we try to satisfy the descent condition of $\overline{DL+}$ method using the MTTHS direction in (8)-(9), [28]. For this purpose, consider the following subproblem:

$$\min \| d_{k+1}^{\overline{DL+}} - d_{k+1}^{MTTHS} \| \tag{26}$$

Using simple algebraic calculations, we get the $\overline{DL+}$ parameter as following:

$$t_{k_1}^{\overline{DL+}*} = \frac{1}{a_2}(a_1 - a_3 + \frac{a_4}{\|d_k\|^2} d_k^T z_k), \tag{27}$$

where

$$a_1 = \frac{g_{k+1}^T \overline{y}_k}{d_k^T \overline{y}_k}, \qquad a_2 = \frac{g_{k+1}^T s_k}{d_k^T \overline{y}_k}$$
$$a_3 = \frac{g_{k+1}^T z_k}{d_k^T z_k}, \qquad a_4 = \frac{g_{k+1}^T d_k}{d_k^T z_k},$$

where z_k and $\overline{y}_k$ are defined in Eqns (10) and (21), respectively. After some simplification, the Eqn(27) can be written as follows:

$$t_{k_1}^{\overline{DL+}*} = -\frac{\overline{y}_k^T g_{k+1}}{s_k^T g_{k+1}} \tag{28}$$

However, the parameter $t_{k_1}^{\overline{DL+}*}$ should be nonnegative. So, we use the following modified form of this parameter given:

$$t_{k_1}^{\overline{DL+}*} = max\{t_{k_1}^{\overline{DL+}*}, 0\} \tag{29}$$

So, by replacing (29) in (14), we get a new nonlinear $\overline{DL}$ direction as following:

$$d_{k+1}^{NDL-1} = -g_{k+1} + (\frac{g_{k+1}^T y_k}{d_k^T y_k} - t_{k_1}^{\overline{DL+}*} \frac{g_{k+1}^T s_k}{d_k^T y_k})d_k, \tag{30}$$

where $t_{k_1}^{\overline{DL+}*}$ is defined in Eqn (29). The CG method based on the search direction d_{k+1}^{NDL-1}, called "NDL-1" method.

2.2.2 QN approach

Since QN methods apply the second derivative information in search directions, so they are useful in solving large scale unconstrained optimization problems. Therefore, to access the CG direction matrices to approximate the inverse Hessian matrix, similar to [3] in the QN method, we enhance the efficiency of CG method. For this reason, we approach the matrix direction of the $\overline{DL+}$ method, $Q_{k+1}^{\overline{DL+}}$, to the Shanno quasi-Newton direction matrix, Q_{k+1}^S, Eqn (13). Therefore, Consider the following subproblem:

$$t_{k_2}^{\overline{DL+}*} = argmin\|Q_{k+1}^{\overline{DL+}} - Q_{k+1}^S\|_F, \tag{31}$$

where $\|.\|_F$ is Frobenius norm. Using the property $tr(AA^T) = \|A\|_F^2$ and after some algebraic calculations, we have

$$t_{k_2}^{\overline{DL+}*} = 1 + \frac{\overline{y}_k^T \overline{y}_k}{s_k^T \overline{y}_k} - \frac{s_k^T \overline{y}_k}{\|s_k\|^2} \tag{32}$$

Now, similar to (29), we propose the following $\overline{DL}$ parameter:

$$t_{k_2}^{\overline{DL+}*} = \{t_{k_2}^{\overline{DL+}*}, 0\} \tag{33}$$

So, by replacing (33) in (14), we get another new $\overline{DL+}$ direction as following:

$$d_{k+1}^{NDL-2} = -g_{k+1} + (\frac{g_{k+1}^T \overline{y}_k}{d_k^T \overline{y}_k} - t_{k_2}^{\overline{DL+}*} \frac{g_{k+1}^T s_k}{d_k^T \overline{y}_k})d_k, \tag{34}$$

where $t_{k_2}^{\overline{DL+*}}$ is defined in Eqn (33). The CG method based on d_{k+1}^{NDL-2}, called "NDL-2" method. Now, we discuss the global convergence of the "NDL-1" and "NDL-2" methods. So, we need to make the following underlying assumptions on the objective function, commonly used in the convergence analysis of the CG methods [34].

Assumption (A):

Let the objective function f is strongly convex and ∇f is Lipschitz continuous on the level set

$$S = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\} \tag{35}$$

That is there exists constants $\mu > 0$ and L such that

$$(\nabla f(x) - \nabla f(y))^T (x - y) \geq \mu \|x - y\|^2, \quad \forall x, y \in S \tag{36}$$

and

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad \forall x, y \in S \tag{37}$$

From Eqns (36)-(37), there exists a positive constant Γ such that for all $x \in S$; $\|\nabla f(x)\| \leq \Gamma$.

Lemma 1. [30] Let the Assumption (A) holds. Consider any CG method in the form of (2)-(3) in which for all $k \geq 0$, the search direction d_k is a descent direction, and the step length α_k is determined to satisfy the Wolfe conditions, (4)-(5). If

$$\sum_{k \geq 0} \frac{1}{\|d_k\|^2} = \infty \tag{38}$$

then the method converges in the sense that

$$\liminf_{k \to \infty} \|g_k\| = 0 \tag{39}$$

Theorem 1. Let the Assumption (A) holds for the objective function f in (1). Consider a CG method in the form of (2)-(3) with the CG direction defined by (30), "NDL-1" method, in which the step length α_k is computed such that the Wolfe conditions (4)-(5) are satisfied. If the objective function f is uniformly convex on $\mathcal{S}$, then the method converges in the sense that (39) holds.

Proof. For any uniform convex differentiable function f, there exists a positive constant μ such that (see Theorem 1.3.16 of [30])

$$y_k^T s_k \geq \mu \|s_k\|^2 \tag{40}$$

Also similar inequality can be proved by replacing y_k with $\overline{y}_k$. For this purpose we have

$$\overline{y}_k^T s_k = (y_k + \frac{\bar{\theta}_k}{s_k^T u_k} u_k)^T s_k = s_k^T y_k + max\{\theta_k, 0\} \geq s_k^T y_k \geq \mu \|s_k\|^2 \tag{41}$$

Note that, from the second equation of the Wolf conditions, Eqn (4), we have:

$$g_{k+1}^T d_k \geq \sigma g_k^T d_k \tag{42}$$

On the other hand, from (21) we have:

$$\|\overline{yk}_k\| \leq \|y_k\| + \|A_k u_k\| = \|y_k\| + \|w_k\| \tag{43}$$

where $w_k = A_k u_k$ and A_k is defined in (21). Now we show that $\|\overline{y}_k\| \leq L_1\|s_k\|$. For this purpose, first of all, using Taylor expansion of θ_k in Eqn (22), we have:

$$|\theta_k| < M\|s_k\|^2 \tag{44}$$

Then, considering the Eqn(21), we have two cases for w_k, [10]: $w_k = \frac{\theta_k s_k}{\|s_k\|^2}$ or $w_k = \frac{\theta_k y_k}{s_k^T y_k}$, which θ_k is defined in (21). In the first case, from Eqns (37), (43) and (44),we get:

$$\|\overline{y}_k\| \leq \|y_k\| + \frac{|\theta_k|\|s_k\|}{\|s_k\|^2} = (L+M)\|s_k\| = M_1\|s_k\|, \tag{45}$$

where $M_1 = L + M$. In the second case, from (37), (40) and (44) we have:

$$\|\overline{y}_k\| \leq \|y_k\| + \frac{ML\|s_k\|^3}{\mu\|s_k\|^2} \leq L(1+\frac{M}{\mu})\|s_k\| = M_2\|s_k\|, \tag{46}$$

where $M_2 = L(1+\frac{M}{\mu})$. Now, let $L_1 = max\,\{M_1, M_2\}$, then we have:

$$\|\overline{y}_k\| \leq L_1\|s_k\| \tag{47}$$

Next we can show that $\|z_k\| \leq L_2\|s_k\|$, where z_k is defined in (10). From the eqns (10) and (37), we have:

$$\begin{aligned}\|z_k\| = \|y_k + c\|g_k\|^r s_k\| &\leq \|y_k\| + c\|g_k\|^r\|s_k\| \leq L\|s_k\| + c\|g_k\|^r\|s_k\| \\ &\leq (L + c\Gamma^r)\|S_k\| = L_2\|s_k\|,\end{aligned} \tag{48}$$

where $L_2 = L + c\Gamma^r$. Moreover, from (40) and (10) we have:

$$\begin{aligned}s_k^T z_k = s_k^T(y_k + c\|g_k\|^r s_k) &= s_k^T y_k + c\|g_k\|^r\|s_k\|^2 \\ &\geq (\mu + c\|g_k\|^r)\|s_k\|^2 \geq \mu\|s_k\|^2,\end{aligned} \tag{49}$$

which implies that $s_k^T z_k \geq \mu\|s_k\|^2$. Hence from this inequality and Eqns (41), (47), (48), (49), (5) and Cauchy-Shwartz inequality we have:

$$\begin{aligned}|t_{k_1}^{\overline{DL}*}| &= \frac{d_k^T\overline{y}_k}{g_{k+1}^T s_k}\left(\frac{g_{k+1}^T\overline{y}_k}{d_k^T\overline{y}_k} + \frac{g_{k+1}^T z_k}{s_k^T z_k} + \frac{g_{k+1}^T d_k}{\|d_k\|^2}\right) \\ &\leq \frac{\|s_k\|\|\overline{y}_k\|}{\sigma\|g_{k+1}\|\|s_k\|}\left(\frac{\|g_{k+1}\|\|\overline{y}_k\|}{\overline{\mu}\|s_k\|^2} + \frac{\|g_{k+1}\|\|z_k\|}{\mu\|s_k\|^2} + \frac{\|g_{k+1}\|\|s_k\|}{\|s_k\|^2}\right) \\ &\leq \frac{L_1}{\sigma}\left(\frac{L_1}{\overline{\mu}} + \frac{L_2}{\mu} + 1\right)\end{aligned} \tag{50}$$

That is $t_{k_1}^{\overline{DL+*}}$ is bounded for uniformly convex objective function. So, if we use the Wolfe conditions, (4)-(5), similar to Theorem (2.1) in [25], the search directions are bounded away, which with Lemma 1 complete the proof. □

Theorem 2. Let Assumption (A) holds for the objective function f in (1). Consider a CG method in the form of (2)-(3) with the CG direction defined by (34), "NDL-2" method, in which the step length α_k is computed such that the Wolfe conditions (4) and (5) are satisfied. If the objective function f is uniformly convex on $\mathcal{S}$, then the method converges in the sense that (39) holds.

Proof. Considering the Assumption (A) and the assumptions of Theorem 1, from eqns (36), (41), (45), (47) and definition of $t_{k_2}^{\overline{DL+}*}$, Eqn (33), we have:

$$|t_{k_2}^{\overline{DL+}*}| = |1 + \frac{\overline{y}_k^T \overline{y}_k}{s_k^T \overline{y}_k} - \frac{s_k^T \overline{y}_k}{\|s_k\|^2}| \leq 1 + \frac{\|\overline{y}_k\|^2}{|s_k^T \overline{y}_k|} + \frac{|s_k^T \overline{y}_k|}{\|s_k\|^2}$$

$$\leq 1 + \frac{L_1^2 \|s_k\|^2}{\|s_k\|^2} + \frac{\|s_k\| \|\overline{y}_k\|}{\|s_k\|^2} = 1 + L_1 + \frac{L_1^2}{\mu} \tag{51}$$

So, similar to Theorem 1, the search directions are bounded away, and the proof is complete. □

In order to ensure the global convergence of the proposed CG methods, "NDL-1" and "NDL-2" methods, for general functions, we modify the CG parameter in Eqn (24), similar to [35, **?**], as follows:

$$\beta_{k_i}^{\overline{DL}+} = max\{\frac{g_{k+1}^T \overline{y}_k}{d_k^T \overline{y}_k}, 0\} - t_{k_i}^{\overline{DL+}*} \frac{g_{k+1}^T s_k}{d_k^T \overline{y}_k}, \ i = 1, 2 \tag{52}$$

where $t_{k_i}^{\overline{DL+}*}$, $i = 1, 2$ is defined in (29) and (33), respectively. Theorem 3.6 of [35] ensures the global convergence of the methods, which are named $\overline{DL}+$, for general functions, if the search directions satisfy the sufficient descent condition.

3 Numerical Experiments

In this section, we present some numerical experiments, obtained by applying a MATLAB 8.8.0.1 (R2013a) implementation of the proposed nonlinear CG methods, "NDL-1" and "NDL-2". The numerical results are compared with the $DL+$ [35] with parameter $t = 0.1$ and DK [33] with parameter $\tau_k = \frac{\|y_k\|^2}{s_k^T y_k}$. We perform the implementations on a computer, Intel(R) Core (TM) A10-8700P CPU 3.20 Gigahertz 64-bit desktop with 8 Gigabyte RAM. Our experiments have been done on a set of test problems of unconstrained optimization problems of CUTEr collection [1]. Although the descent property may not always hold for the proposed method, the upward search direction seldom occurred in our experiments; when encountering, we restarted the algorithm with Powell Restart [30], which is $|g_k^T g_{k+1}| < 0.2\|g_{k+1}\|$.

Moreover, we used the active approximate Wolfe conditions described in (4)-(5) with parameters $\sigma = 0.9$ and $\rho = 10^{-4}$. The same stop condition is considered for all methods, which are $\|g_k\|_\infty \leq 10^{-6}$ and the maximum number of iterations is limited to 1000. Table 1, shows our comparing data contains the test problems, dimensions (n), the total number of function evaluations (f_n) and the total number of gradient evaluations (g_n), respectively.

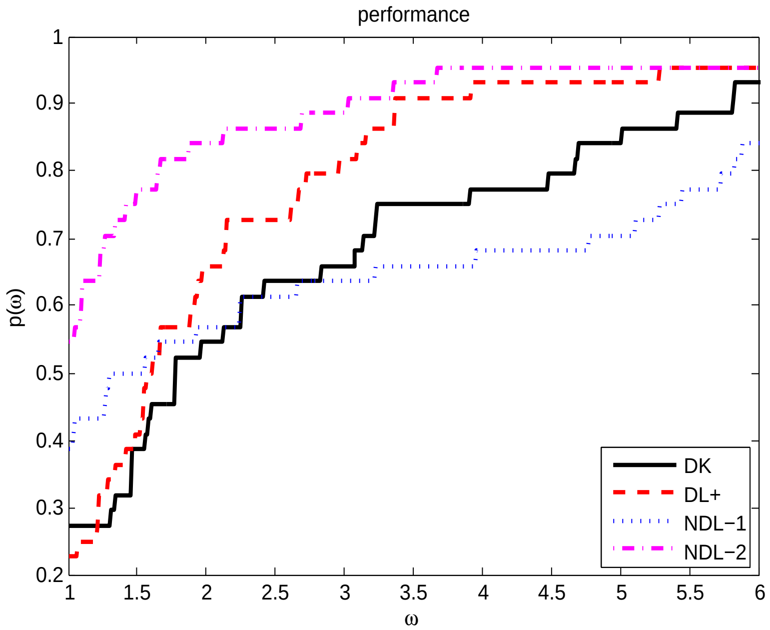

Figure 1: Performance profiles based on the number of iterations for "NDL-1", "NDL-2", DL+ and DK methods.

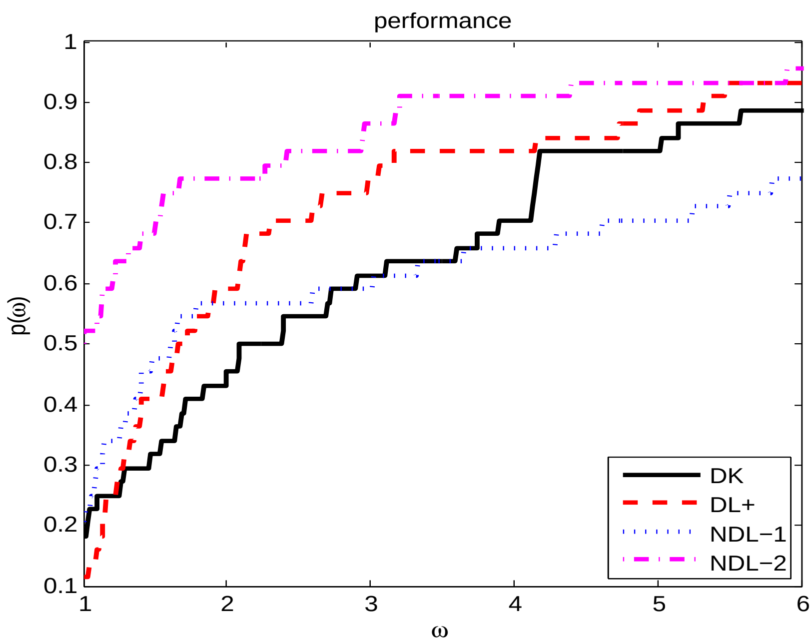

Figure 2: Performance profiles based on CPU time for "NDL-1", "NDL-2", DL+ and DK methods.

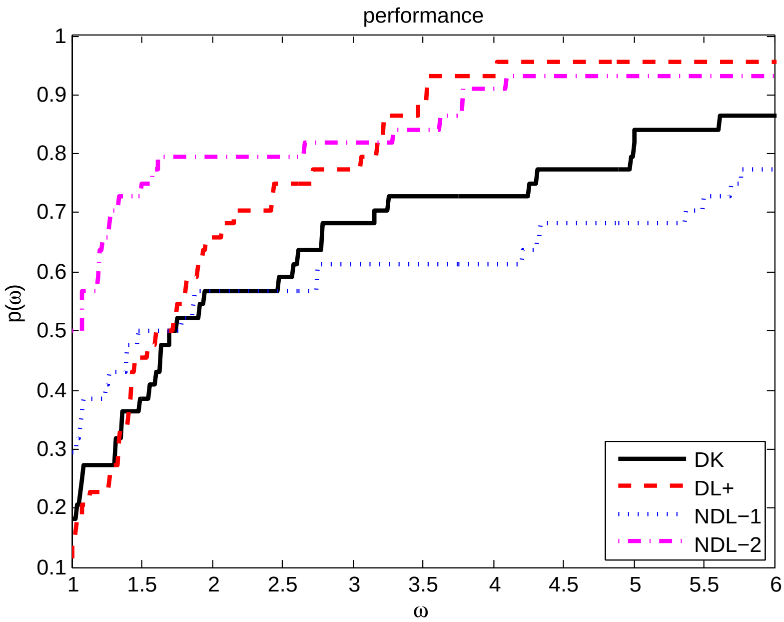

Figure 3: Performance profiles based on $n_f + 3n_g$ for "NDL-1", "NDL-2", DL+ and DK methods.

For more comparison on our numerical results, we apply the performance profile introduced by Dolan and More´ [8].

Table 1: Experiments results of the proposed methods about the total function evaluations (f_n) and gradient evaluations (g_n)

		$DL+$	DK	$NDL-1$	$NDL-2$
Problem	n	$f_n \setminus g_n$	$f_n \setminus g_n$	$f_n \setminus g_n$	$f_n \setminus g_n$
AKIVA	2	2 \ 2	2 \ 2	2 \ 2	2 \ 2
ALLINITU	4	451 \ 313	626 \ 408	622 \ 404	421 \ 326
ARGLINA	200	17 \ 17	18 \ 18	7 \ 7	11 \ 11
ARGLINB	200	45267 \ 2003	33853 \ 766	33853 \ 766	41262 \ 1258
ARGLINC	200	76960 \ 3407	153903 \ 3479	153903 \ 3479	35670 \ 1081
ARWHEAD	5000	8136 \ 1077	36169 \ 3122	71908 \ 6014	668 \ 7225 \ 669
BARD	3	6018 \ 2672	23350 \ 8756	22325 \ 8326	4163 \ 1749
BDQRTIC	5000	19057 \ 1820	142535 \ 10001	143643 \ 10001	22755 \ 1717
BEALE	2	3490 \ 1413	2753 \ 949	2719 \ 946	990 \ 421
BIGGS6	6	4271 \ 3784	1670 \ 1319	7928 \ 7719	530 \ 429
BOX	10000	11453 \ 1123	115065 \ 10001	119587 \ 10001	6361 \ 587
BOX3	3	60 \ 59	29 \ 28	54 \ 53	1016 \ 998
BRKMCC	2	455 \ 179	3087 \ 965	3103 \ 970	1496 \ 599
BROWNAL	200	85271 \ 10001	122892 \ 8870	139672 \ 10001	24583 \ 1892
BROWNDEN	4	23116 \ 1816	70244 \ 5015	29466 \ 2129	16501 \ 1286

Continued on next page

Table 1 – *Continued from previous page*

		$DL+$	DK	$NDL-1$	$NDL-2$
$Problem$	n	$f_n \setminus g_n$	$f_n \setminus g_n$	$f_n \setminus g_n$	$f_n \setminus g_n$
$BROYDN7D$	5000	34827 \ 7796	49687 \ 10001	48910 \ 10001	26146 \ 6441
$BRYBND$	5000	4336 \ 1063	1845 \ 495	1779 \ 511	2659 \ 702
$CHAINWOO$	4000	21411 \ 2385	54444 \ 6153	91905 \ 10001	16734 \ 1961
$CHNROSNB4$	50	13102 \ 1757	67274 \ 8503	66035 \ 8374	10379 \ 1464
$CHNRSNBM$	50	15446 \ 20934	54485 \ 7030	47455 \ 6218	10953 \ 1571
$CLIFF$	2	32270 \ 10001	10037 \ 10001	13987 \ 10001	37944 \ 10001
$CUBE$	2	3891 \ 739	81088 \ 100014	81353 \ 10001	2893 \ 575
$CURLY10$	10000	104174 \ 10001	108942 \ 10001	108925 \ 10001	100641 \ 10001
$CURLY20$	10000	122180 \ 10001	127249 \ 10001	127199 \ 10001	121138 \ 10001
$CURLY30$	10000	132988 \ 100014	138686 \ 10001	138608 \ 10001	130225 \ 10001
$DECONVU$	63	18656 \ 5170	6881 \ 1898	8575 \ 2421	11110 \ 3554
$DENSCHNA$	2	25 \ 25	33 \ 33	26 \ 26	27 \ 27
$DENSCHNB$	2	16 \ 16	17 \ 17	10 \ 10	12 \ 12
$DENSCHNC$	2	1642 \ 651	2146 \ 1417	3643 \ 1164	867 \ 439
$DENSCHND$	3	2354 \ 283	741 \ 96	100410 \ 3841	3038 \ 288
$DENSCHNE$	3	21 \ 18	21 \ 18	12 \ 9	16 \ 13
$DENSCHNF$	2	6528 \ 1158	2136 \ 375	12319 \ 2203	5669 \ 1006
$DIXMAANC$	3000	20 \ 184	19 \ 17	14 \ 12	15 \ 13
$DIXMAANA$	3000	17 \ 16	18 \ 17	11 \ 10	13 \ 12
$DIXMAANB$	3000	19 \ 18	18 \ 17	11 \ 10	14 \ 13
$DIXMAANC$	3000	20 \ 18	19 \ 17	14 \ 12	15 \ 13
$DIXMAAND$	3000	609 \ 76	21 \ 17	16 \ 12	3717 \ 289
$DIXMAANE$	3000	279 \ 278	1313 \ 1312	1093 \ 1092	143 \ 142
$DIXMAANF$	3000	760 \ 759	625 \ 624	393 \ 392	496 \ 495
$DIXMAANG$	3000	219 \ 217	425 \ 423	276 \ 274	794 \ 792
$DIXMAANH$	3000	9281 \ 863	13364 \ 1332	157256 \ 10001	5123 \ 553
$DIXMAANI$	3000	302 \ 287	117 \ 116	75 \ 74	546 \ 156
$DIXMAANJ$	3000	1191 \ 1188	580 \ 579	367 \ 366	630 \ 629
$DIXMAANK$	3000	322 \ 320	414 \ 412	254 \ 252	626 \ 624
$DIXMAANL$	3000	107360 \ 10001	6648 \ 616	136154 \ 10001	147794 \ 10001
$DIXMAANM$	15	231 \ 231	172 \ 172	810 \ 810	199 \ 199
$DIXMAANN$	15	207 \ 206	1299 \ 1298	754 \ 753	174 \ 173
$DIXMAANO$	15	203 \ 201	1309 \ 1307	747 \ 745	171 \ 169
$DIXMAANP$	15	194 \ 191	1309 \ 1306	742 \ 739	176 \ 173
$DIXON3DQ$	10000	208 \ 208	286 \ 281	1218 \ 1218	1003 \ 1003
$DJTL$	2	13213 \ 606	36052 \ 1461	30600 \ 1246	11650 \ 532
$DQDRTIC$	5000	11776 \ 2069	27788 \ 4556	27788 \ 4556	8663 \ 1591
$DQRTIC$	5000	13757 \ 843	8647 \ 693	5849 \ 398	14899 \ 993
$EDENSCH$	2000	3449 \ 797	3215 \ 1299	3762 \ 1361	3509 \ 1182

Continued on next page

Table 1 – *Continued from previous page*

		$DL+$	DK	$NDL-1$	$NDL-2$
Problem	n	$f_n \setminus g_n$	$f_n \setminus g_n$	$f_n \setminus g_n$	$f_n \setminus g_n$
$EG2$	1000	1592 \ 358	8142 \ 866	6019 \ 799	1891 \ 363
$ENGVAL1$	5000	2434 \ 1092	1308 \ 574	641 \ 241	1535 \ 677
$ENGVAL2$	3	9163 \ 1069	104609 \ 10001	104593 \ 10001	6510 \ 807
$ERRINROS$	50	83940 \ 10001	71123 \ 10001	70231 \ 10001	74704 \ 10001
$ERRINRSM$	50	81101 \ 10001	82952 \ 10001	77688 \ 10001	83878 \ 10001
$EXPFIT$	2	4474 \ 1021	10474 \ 1730	11086 \ 1838	2818 \ 633
$EXTROSNB$	1000	76398 \ 10001	19469 \ 2277	49393 \ 5073	71250 \ 10001
$FLETBV3M$	5000	1062 \ 1062	328 \ 328	1149 \ 1149	136 \ 136
$FLETCBV3$	5000	10001 \ 10001	10041 \ 10001	10001 \ 10001	10017 \ 10001
$FLETCHBV$	5000	10001 \ 10001	10004 \ 10001	10001 \ 10001	10001 \ 10001
$FLETCHCR$	1000	74350 \ 9375	83444 \ 10001	83284 \ 10001	75752 \ 10001
$FMINSRF2$	5625	620 \ 620	402 \ 402	3670 \ 3670	399 \ 399
$FMINSURF$	5625	622 \ 622	485 \ 485	3678 \ 3678	443 \ 443
$FREUROTH$	5000	16915 \ 1829	77020 \ 9110	97310 \ 10001	11725 \ 1307
$GENHUMPS$	5000	70178 \ 10001	73098 \ 10001	71091 \ 10001	67735 \ 10001
$GENROSE$	500	26425 \ 3337	84050 \ 10001	83962 \ 10001	32775 \ 4311
$GULF$	3	36993 \ 10001	57851 \ 10001	59339 \ 10001	55008 \ 10001
$HAIRY$	2	12791 \ 1469	8686 \ 971	9463 \ 1074	10611 \ 1287
$HATFLDD$	3	15526 \ 10001	14806 \ 10001	11764 \ 7673	23089 \ 10001
$HATFLDE$	3	130 \ 124	1469 \ 819	1800 \ 1206	33097 \ 10001
$HATFLDFL$	3	769 \ 290	1460 \ 487	1370 \ 457	716 \ 180
$HEART6LS$	6	133006 \ 9038	113591 \ 10001	118906 \ 10001	114234 \ 7332
$HEART8LS$	8	10175 \ 1443	84531 \ 10001	82921 \ 10001	12482 \ 1764
$HELIX$	3	8600 \ 1268	14178 \ 2131	11806 \ 1781	7906 \ 1207
$HIELOW$	3	2 \ 2	2 \ 2	2 \ 2	2 \ 2
$HILBERTA$	2	159 \ 159	315 \ 315	181 \ 181	67 \ 67
$HILBERTB$	10	126 \ 110	291 \ 249	291 \ 249	289 \ 263
$HIMMELBB$	2	41 \ 27	107 \ 93	74 \ 60	32 \ 18
$JENSMP$	2	70 \ 10	117 \ 13	13111 \ 857	4212 \ 497
$KOWOSB$	4	81 \ 81	99 \ 80	141 \ 141	27 \ 25
$LIARWHD$	5000	4561 \ 573	126596 \ 10001	129301 \ 10001	31873 \ 2659
$LOGHAIRY$	2	331 \ 331	1748 \ 1748	10001 \ 10001	10001 \ 10001
$MANCINO$	100	35018 \ 1796	196345 \ 10001	196351 \ 10001	35270 \ 1811
$MARATOSB$	2	4201 \ 375	11849 \ 579	12329 \ 602	9751 \ 807
$MEXHAT$	2	18659 \ 1080	62163 \ 2941	77494 \ 3640	12668 \ 738
$NONCVXU2$	5000	18932 \ 10001	27496 \ 10001	27503 \ 10001	15803 \ 10001
$NONCVXUN$	5000	20774 \ 10001	32200 \ 10001	32296 \ 10001	19306 \ 10001
$NONDQUAR$	5000	1341 \ 309	1556 \ 692	2318 \ 1049	1556 \ 378
$OSBORNEA$	5	408 \ 45	135339 \ 10001	133752 \ 10001	24 \ 4

Continued on next page

Table 1 – *Continued from previous page*

		$DL+$	DK	$NDL-1$	$NDL-2$
$Problem$	n	$f_n \setminus g_n$	$f_n \setminus g_n$	$f_n \setminus g_n$	$f_n \setminus g_n$
$PALMER1C$	8	209967 \ 10001	259169 \ 10001	259585 \ 10001	240573 \ 10001
$PALMER2C$	8	173444 \ 10001	223136 \ 10001	223115 \ 10001	202956 \ 10001
$PALMER3C$	8	156749 \ 10001	205456 \ 10001	205364 \ 10001	185655 \ 10001
$PALMER4C$	8	156752 \ 10001	205456 \ 10001	205364 \ 10001	185582 \ 10001
$PALMER5C$	6	2687 \ 1250	1079 \ 450	1091 \ 455	1656 \ 859
$PALMER6C$	8	126013 \ 10001	169991 \ 10001	170001 \ 10001	152097 \ 10001
$PALMER8C$	8	128521 \ 10001	173751 \ 10001	173697 \ 10001	154526 \ 10001
$HIMMELBG$	2	10 \ 10	13 \ 7	7 \ 7	17 \ 12
$HIMMELBH$	2	16 \ 16	16 \ 16	11 \ 11	23 \ 23
$POWELLSG$	5000	4253 \ 1084	39270 \ 7596	34586 \ 6692	3560 \ 870
$POWER$	10000	35680 \ 1839	71636 \ 7038	104337 \ 10001	30141 \ 1621
$QUARTC$	5000	13757 \ 843	8647 \ 693	5849 \ 398	14899 \ 993
$ROSENBR$	2	4068 \ 875	83000 \ 10001	83733 \ 10001	2244 \ 491

Figure 1, to the number of iteration, and Figure 2, to the running time, shows that the ”NDL-2” method slightly outperforms the ”NDL-1”, $DL+$ and the DK methods. In addition, Figure 3 shows that to the $n_f + 3n_g$, the ”NDL-2” method is competitive with the $DL+$ method.

4 Conclusion

Here, using DL approach, we provide a new conjugacy condition by a modified secant equation proposed in [10]. To set the parameter of the new conjugacy condition, $\overline{DL+}$ parameter, two approaches are used. The convergence analysis is presented for uniformly convex and general nonlinear functions. The comparison of the new nonlinear CG methods with some well-known methods, shows that ”NDL-2” method is better in the iteration criteria and in CPU time, although to the $n_f + 3n_g$ is comparative with $DL+$ method.

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Control and Optimization in Applied Mathematics (COAM)
DOI. 10.30473/coam.2019.46226.1115
Vol. 3, No. 2, Autumn-Winter 2018(77-96), ©2016 Payame Noor University, Iran

A Fully Fuzzy Method of Network Data Envelopment Analysis for Assessing Revenue Efficiency Based on Ranking Functions

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Received: May 15, 2019; **Accepted:** November 30, 2019.

Abstract. The purpose of this paper is to evaluate the revenue efficiency in the fuzzy network data envelopment analysis. Precision measurements in real-world data are not practically possible, so assuming that data is crisp in solving problems is not a valid assumption. One way to deal with imprecise data is fuzzy data. In this paper, linear ranking functions are used to transform the full fuzzy efficiency model into a precise linear programming problem and, assuming triangular fuzzy numbers, the fuzzy revenue efficiency of decision makers is measured. In the end, a numerical example shows the proposed method.

Keywords. Network data envelopment analysis, Revenue efficiency, Full fuzzy linear programming, Ranking function.

MSC. 90C34; 90C40.

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1 Introduction

Data envelopment analysis (DEA), initially developed by [3], is a non-parametric technique for evaluating the relative efficiencies of homogeneous decision-making units (DMUs) in terms of multiple inputs and multiple outputs. The basic DEA models and their numerous theoretical and methodological extensions have been reported in [6]. Unlike the black box model, the Network Data Envelopment Analysis (NDEA) model considers all internal processes in performance evaluation. For example, many companies are composed of several sections that have linked activities such as Figure 1. In this example, the company has 3 sections. Each section uses its input resources to generate its output. In either case, there are links or intermediate products that are shown by the link $1 \longrightarrow 2$ and $1 \longrightarrow 3$, and the link $2 \longrightarrow 3$. The link $1 \longrightarrow 2$ shows that part of the outputs of section 1 are used as inputs in section 2. In the current DEA models, each activity must belong to an input or output, and not both, so these models cannot be formulated with intermediate products. For the first time in the year 2000, Fare and

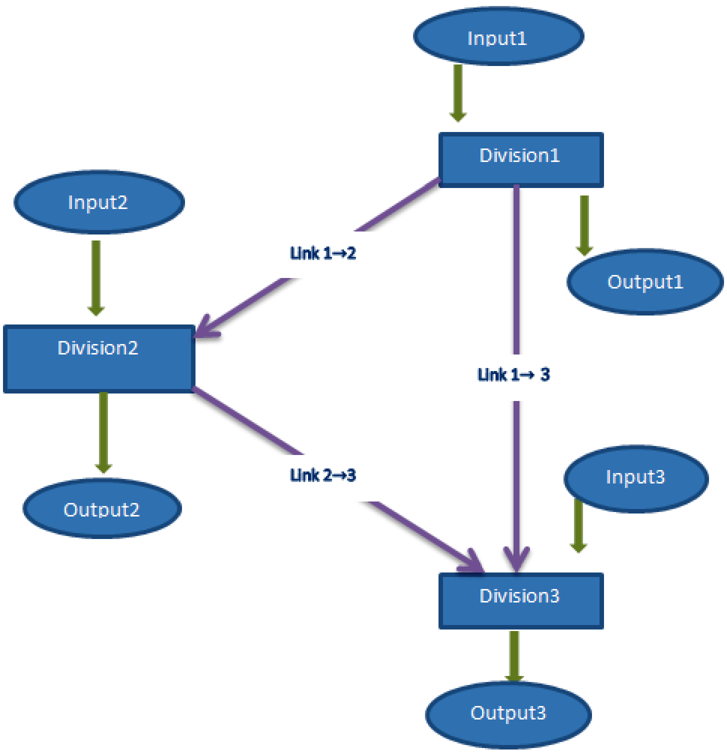

Figure 1: A company with three linked activities

Grosskopf [7] provided network data envelopment analysis models. Their models were expanded by several authors. Sexton and Lewis presented a multi-stage network data envelopment analysis model in 2004 as an extension of the Lewis and Sexton two-step data envelopment analysis model [9]. This article solves a dea model independently for each NODE. Tone and Tsutsui [16] presented a network-based data envelopment analysis model in 2009 based on the SBM model.

The Revenue Efficiency Model (RE) seeks to find a unit that receives the highest revenue from inputs equal to the inputs of the unit under consideration, from the sales of non-less than the outputs of the unit under evaluation. Revenue Efficiency is defined as the ratio of observed revenue to the maximum possible revenue. Given the fact that in the real world we are dealing with network data envelopment analysis, it is important for managers to evaluate the revenue efficiency in NDEA. In 2013, Bani Hashemi and Tohidi [2] presented a model for assessing the revenue efficiency of network data envelopment analysis models.

Classical DEA models assume that all data is crisp. However, crisp data is not always available because the nature of data can be vague and unclear. In this case, one of the important methods for dealing with inaccurate data is to consider fuzzy data. Only in [12] and [13] the fuzzy revenue efficiency (FRE) with input- outputs fuzzy and fuzzy input prices is discussed. Aghayi [1] is examined revenue efficiency measurement with undesirable data in fuzzy DEA and also Kordrostami and Jahani Sayyad Noveiri [8] are studied fuzzy revenue efficiency in sustainable supply chains.

However, in none of these studies, the measurement of fuzzy revenue efficiency has not been mentioned in Full Fuzzy Network Data Envelopment Analysis (FFNDEA). In this paper, we examine full-fuzzy models of network data envelopment analysis (fuzzy input-outputs and fuzzy input prices) to evaluate fuzzy revenue efficiency. Here, the method of ranking functions is used. Therefore, the ranking functions transform the full fuzzy model of network revenue efficiency into a crisp linear programming problem for measuring the fuzzy network revenue efficiency. The rest of the article will be as follows. In section 2, we refer to fuzzy clauses. In the next section, the problem of fuzzy linear programming and its transformation into a crisp problem is studied. section 4 addresses the measurement of revenue efficiency in the DEA, and in Sections 5 and 6 is examined network data envelopment analysis based on SBM model and revenue efficiency in it. Section 7,the proposed method for measuring fuzzy revenue efficiency in FFNDEA is presented and, based on the proposed method, a numerical example is solved in the last section.

2 Fuzzy Premises

2.1 Basic Definitions of Fuzzy

In this section, the basic definitions and the symbols of the fuzzy sets [17, 18], fuzzy Numbers [4], Ranking function [10], and the FFLP concept used in this article.

Definition 1. [17] A fuzzy set $\tilde{A}$ is defined in the reference set X with $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ where $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$ is the membership function and $\mu_{\tilde{A}}(x)$ is the degree of x in A.

Definition 2. [18] Regarding X as the reference set, then fuzzy set A will be convex if and only if for every $x_1, x_2 \in X$:

$$\mu_{\tilde{A}}\big(\lambda x_1 + (1 - \lambda)x_2\big) \geq \min\big(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\big) \qquad \forall \lambda \in [0, 1]$$

Definition 3. [18] Assuming that X is the reference set, then the fuzzy set A is called normal provided that there exist $x \in X$ so that $\mu_{\tilde{A}}(x) = 1$.

Definition 4. [18] A fuzzy number $\tilde{A}$ is a convex normalized fuzzy set $\tilde{A}$ of the real line R such that

1. it exists exactly one $x_0 \in R$ $\mu_{\tilde{A}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{A}$).
2. $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Definition 5. [18] A triangular fuzzy number (TFN), $\tilde{A} = (a^l, a^m, a^u)$ is a fuzzy number with the given membership function $\mu_{\tilde{A}}$

$$\mu_{\tilde{A}}(x) = \begin{cases} (x - a^l)/(a^m - a^l) & a^l < x \leq a^m \\ (x - a^u)/(a^m - a^u) & a^m \leq x < a^u \\ 0 & otherwise. \end{cases}$$

Definition 6. A triangular fuzzy number $\tilde{A} = (a^l, a^m, a^u)$ is called a nonnegative number if and only if a $a^l \geq 0,\ a^m - a^l \geq 0,\ a^u - a^m \geq 0$ and it is a positive number if and only if $a^l > 0,\ a^m - a^l \geq 0,\ a^u - a^m \geq 0$.

Definition 7. The support of a fuzzy set $\tilde{A}$, $S(\tilde{A})$ is the crisp set of all $x \in X$ such that . $\mu_{\tilde{A}}(x) > 0$. The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α-cut set: $A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$

Definition 8. [10] Suppose $\mathfrak{F}$ a set of all triangular fuzzy numbers. If $\tilde{A} \in \mathfrak{F}$, $[A^1_\alpha, A^u_\alpha]$, $\alpha \in [0, 1]$ the α- cut is $\tilde{A}$. Then, the ranking function of a function $\mathfrak{R} : \mathfrak{F} \longrightarrow \mathbb{R}$ is:

$$\mathfrak{R}(\tilde{A}) = \frac{1}{2}\int_0^1 (A^l_\alpha + A^u_\alpha)d\alpha$$

If $\tilde{A} = (a^l, a^m, a^u)$ is a triangular fuzzy number, then $\mathfrak{R}(\tilde{A}) = \dfrac{1}{4}(a^l + 2a^m + a^u)$.

Definition 9. [10] If $\tilde{A} = (a^l, a^m, a^u)$ and $\tilde{B} = (b^l, b^m, b^u)$ are two triangular fuzzy numbers, then order of $\tilde{A}$ and $\tilde{B}$ based on the ranking function $\mathfrak{R}$ will be:

(i) $\tilde{A} \preccurlyeq \tilde{B} \Longleftrightarrow \mathfrak{R}(\tilde{A}) \leq \mathfrak{R}(\tilde{B})$

(ii) $\tilde{A} \succcurlyeq \tilde{B} \Longleftrightarrow \mathfrak{R}(\tilde{A}) \geq \mathfrak{R}(\tilde{B})$

(iii) $\tilde{A} \approx \tilde{B} \Longleftrightarrow \mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B})$

And the features of Linearity will be:

$$\mathfrak{R}(k\tilde{A} + \tilde{B}) = k\mathfrak{R}(\tilde{A}) + \mathfrak{R}(\tilde{B}), \qquad k \in \mathbb{R}$$

2.2 Math Operations on Triangular Fuzzy Numbers

If $\tilde{A} = (a^l, a^m, a^u)$ and $\tilde{B} = (b^l, b^m, b^u)$ are two triangular fuzzy numbers, then the mathematical operations on triangular fuzzy numbers will be as follows:

(i)	Addition	$\tilde{A} \oplus \tilde{B} \approx (a^1 + b^1, a^m + b^m, a^u + b^u)$
(ii)	Subtraction	$\tilde{A} \ominus \tilde{B} \approx (a^1 - b^u, a^m - b^m, a^u - b^1)$
(iii)	Multiplication	$\tilde{A} \otimes \tilde{B} \approx (a^1 b^1, a^m b^m, a^u b^u), \quad \tilde{A}, \tilde{B} \succcurlyeq \tilde{0}$
(iv)	Division	$\frac{\tilde{A}}{\tilde{B}} \approx \frac{(a^1, a^m, a^u)}{(b^1, b^m, b^u)} \approx \left(\frac{a^1}{b^u}, \frac{a^m}{b^m}, \frac{a^u}{b^1}\right), \quad \tilde{A}, \tilde{B} \succ \tilde{0}$
(v)	Scalar multiplication	$\forall k \in \mathbb{R},\ k\tilde{A} \approx \begin{cases} (ka^1, ka^m, ka^u), & k > 0 \\ (ka^u, ka^m, ka^1), & k < 0 \end{cases}$

3 Fuzzy linear programming problem

A linear programming problem with fuzzy coefficients and variables is called a full fuzzy linear programming problem. A full-fuzzy linear programming problem [11] with m constraints and n fuzzy variables are defined by the following model:

$$\begin{aligned} &\tilde{Z} = \max\,(or\ \min)(\tilde{C}^T \otimes \tilde{X}) \\ &\text{subject to} \quad \tilde{A} \otimes \tilde{X} \preccurlyeq, \approx, \succcurlyeq \tilde{b};\ \tilde{X} \succcurlyeq \tilde{0} \qquad (P1) \end{aligned}$$

where $\tilde{C} = [\tilde{c}_j]_{n\times1}$, $\tilde{X} = [\tilde{x}_j]_{n\times1}$, $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$, $\tilde{b} = [\tilde{b}_j]_{m\times1}$, and $\tilde{a}_{ij}$, $\tilde{c}_j$, $\tilde{b}_i \in \mathfrak{F}$, $\tilde{x}_j$ are non-negative fuzzy numbers and $\tilde{0} = (0, 0, 0)$.

Definition 10. [11] The fuzzy optimal solution to the full-fuzzy linear programming problem $(P1)$ will be $\tilde{X} = [\tilde{x}_j]_{n\times1}$. will apply if the following conditions apply:

1) $\tilde{x}_j$ is a non-negative fuzzy number,

2) $\tilde{A} \otimes \tilde{X} \preccurlyeq, \approx, \succcurlyeq \tilde{b}$,

and 3) If there exist any non-negative fuzzy number such as $\tilde{Y} = [\tilde{y}_j]_{n\times 1}$, to the point where $\tilde{A}\otimes\tilde{X} \preccurlyeq, \approx, \succcurlyeq \tilde{b}$, then $\Re(\tilde{C}^T\otimes\tilde{X}) \geq \Re(\tilde{C}^T\otimes\tilde{Y})$ for the maximization problem and $\Re(\tilde{C}^T\otimes\tilde{X}) \leq \Re(\tilde{C}^T\otimes\tilde{Y})$ for the minimization problem.

Definition 11. [11] Suppose that $\tilde{X} = [\tilde{x}_j]_{n\times 1}$ is the fuzzy optimal solution for full fuzzy linear problem $(P1)$. If there exist any non-negative fuzzy number such as $\tilde{Y} = [\tilde{y}_j]_{n\times 1}$, then $\tilde{A}\otimes\tilde{Y} \preccurlyeq, \approx, \succcurlyeq \tilde{b}$, and $\Re(\tilde{C}^T\otimes\tilde{X}) = \Re(\tilde{C}\otimes\tilde{Y})$, then $\tilde{Y} = [\tilde{y}_j]_{n\times 1}$ is called a fuzzy optimal solution of $(P1)$. Suppose that $\tilde{c}_j = (c_j^1, c_j^m, c_j^u)$, $\tilde{x}_j = (x_j^1, x_j^m, x_j^u)$, $\tilde{a}_{ij} = (a_{ij}^1, a_{ij}^m, a_{ij}^u)$ and $\tilde{b}_j = (b_i^1, b_i^m, b_i^u)$ represents triangular fuzzy numbers. Then, the fuzzy decision parameters and variables in the model $(P1)$ are converted as follows:

$$\begin{aligned}
&\tilde{Z} = \max\,(or\ \min)\Big(\sum_{j=1}^{n}(c_j^1, c_j^m, c_j^u)\otimes(x_j^1, x_j^m, x_j^u)\Big)\\
&\quad\text{subject to}\qquad \sum_{j=1}^{m}(a_{ij}^1, a_{ij}^m, a_{ij}^u)\otimes(x_j^1, x_j^m, x_j^u) \preccurlyeq, \approx, \succcurlyeq\ (b_i^1, b_i^m, b_i^u)\ \forall i;\\
&\quad(x_j^1, x_j^m, x_j^u) \succcurlyeq \tilde{0}\quad \forall j \qquad\qquad (p2)
\end{aligned}$$

After performing the mathematical operations discussed in Section 2-2, the model $(P2)$ is converted to the following form:

$$\begin{aligned}
&\tilde{Z} = \max\,(or\ \min)\ \Big(\sum_{j=1}^{n} c_j^l x_j^l, \sum_{j=1}^{n} c_j^m x_j^m, \sum_{j=1}^{n} c_j^u x_j^u\Big)\\
&\quad\text{subject to}\qquad \Big(\sum_{j=1}^{n} a_{ij}^l x_j^l, \sum_{j=1}^{n} a_{ij}^m x_j^m, \sum_{j=1}^{n} a_{ij}^u x_j^u\Big) \preccurlyeq, \approx, \succcurlyeq (b_i^l, b_i^m, b_i^u)\ \forall i;\\
&\quad(x_j^l, x_j^m, x_j^u) \succcurlyeq \tilde{0}\qquad \forall\ j \qquad\qquad (P3)
\end{aligned}$$

Now, using Nasseri et al.'s algorithm [11] and the ranking method, the FFLP $(P2)$ turns into a precise linear programming problem. The steps in the algorithm are briefly summarized below:
Step 1: Transform full fuzzy objective function using its ranking function $\Big(\Re\big(\sum_{j=1}^n c_j^l x_j^l, \sum_{j=1}^n c_j^m x_j^m, \sum_{j=1}^n c_j^u x_j^u\big)\Big)$ into the crisp format.
Step 2: Full fuzzy constraints of the model $(P2)$ using the following ranking functions are:

$$\begin{aligned}
&\sum_{j=1}^{n} a_{ij}^l x_j^l \leq, =, \geq b_i^1 \qquad \forall i\\
&\sum_{j=1}^{n} a_{ij}^m x_j^m \leq, =, \geq b_i^m \qquad \forall i\\
&\sum_{j=1}^{n} a_{ij}^u x_j^u \leq, =, \geq b_i^u \qquad \forall i
\end{aligned}$$

Step 3: The non-negative Fuzzy constraints, that is, $(x_j^1, x_j^m, x_j^u) \succcurlyeq \tilde{0}\ \forall j$ in the model $(P2)$, which guarantees the decision variables assessment as non-triangular fuzzy numbers, will be as follows:

$$x_j^1 \geq 0,\ x_j^m - x_j^1 \geq 0,\ x_j^u - x_j^m \geq 0,\ \forall j$$

Therefore, using the above steps, the model ($P2$) turns into the exact linear programming problem:

$$
\begin{aligned}
Z = \max\,(or\ \min)\Re\Big(\sum_{j=1}^{n} c_j^l x_j^l, \sum_{j=1}^{n} c_j^m x_j^m, \sum_{j=1}^{n} c_j^u x_j^u\Big) \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij}^l x_j^l \le, =\ge b_i^l \quad \forall i \\
& \sum_{j=1}^{n} a_{ij}^m x_j^m \le, =\ge b_i^m \quad \forall i \qquad\qquad (P4) \\
& \sum_{j=1}^{n} a_{ij}^u x_j^u \le, =\ge b_i^u \quad \forall i \\
& x_j^l \ge 0,\ x_j^m - x_j^l \ge 0,\ x_j^u - x_j^m \ge 0, \quad \forall\, j
\end{aligned}
$$

Theorem 1. Each feasible solution in the model ($P4$) is also a feasible solution in the model ($P3$). Argument in [13].

Theorem 2. The optimal solution of the model ($P4$) is the optimal solution for the model ($P3$) Argument in [13].

4 Revenue Efficiency in DEA

The output-oriented DEA model under the assumption of variable return to scale can be used for calculation of output-oriented technical efficiency and revenue efficiency. Output-oriented model under the assumption of variable return to scale can be written in the following form:

$$
\begin{aligned}
\max \quad & \varphi_0 \\
\text{subject to} \quad & x_{io} \ge \sum_{j=1}^{n} \lambda_j x_{ij}, \qquad i = 1, \cdots, m \\
& \varphi_0 y_0 \le \sum_{j=1}^{n} \lambda_j y_j \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0 \qquad\qquad \forall j
\end{aligned}
$$

Where φ_0 is output-oriented technical efficiency of DMU_o in the output-oriented DEA model. To calculate revenue efficiency the following revenue maximisation DEA problem is necessary to solve [5]:

$$
\begin{aligned}
\max \quad & p_o y \\
\text{subject to} \quad & x_{io} \geq \sum_{j=1}^{n} \lambda_j x_{ij}, \qquad i = 1, \cdots, m \\
& y \leq \sum_{j=1}^{n} \lambda_j y_j \\
& \lambda_j \geq 0 \qquad\qquad\qquad \forall j
\end{aligned}
$$

Where p_o is vector output prices for DMU_o. The overall revenue efficiency is defined as the ratio of observed revenue to maximum revenue for the DMU_o [5]:

$$\alpha^* = p_o y_o / p_o y_o^*$$

where y_o^* is an optimal solution for model [*Revenue*].

4.1 Single output case

In this section, we deal with n DMUs with m inputs $\boldsymbol{x} = (x_1, x_2, \cdots, x_m)$ to produce one output of $y(> 0)$. For a $DMU_o(o = 1, \cdots, n)$, let the inputs and output be $\boldsymbol{x_o} = (x_{1o}, x_{2o}, \cdots, x_{mo})$ and y_o (> 0)respectively, and the unit price of output y_o be p_o (> 0).

Between the two efficiency measures (technical efficiency φ^* and revenue efficiency α^*) we have the following theorem.

Theorem 3. For the single output case, $\alpha^* = 1/\varphi^*$.

Proof. Let us denote y as φy_o in [Revenue] and change the variable from y to φy_o. Then, noting $y_o > 0$ and $p_o > 0$, [Revenue] becomes:

$$
\begin{aligned}
\max \quad & p_o \varphi y_o \\
\text{subject to} \quad & x_{io} \geq \sum_{j=1}^{n} \lambda_j x_{ij}, \qquad i = 1, \cdots, m \\
& \varphi y_o \leq \sum_{j=1}^{n} \lambda_j y_j \\
& \lambda_j \geq 0 \qquad\qquad\qquad \forall j
\end{aligned}
$$

This program is equivalent to [CCR] and its optimal objective value is $\varphi^* p_o y_o$. Thus we have

$$\alpha^* = \frac{p_o y_o}{\varphi^* p_o y_o} = \frac{1}{\varphi^*}$$

□

Definition 12. (Allocative efficiency): The allocative efficiency γ^* of DMU_o is defined as the ratio of revenue efficiency to technical efficiency, ie, $\gamma^* = \frac{\alpha^*}{\varphi^*}$. The allocative efficiency γ^* is less than or equal to one, and DMU_o is called allocatively efficient when $\gamma^* = 1$ holds

4.2 General case

Here we observe a more general case where we have m inputs $\boldsymbol{x} = (x_1, x_2, \cdots, x_m)$ and s outputs $\boldsymbol{y} = (y_1, y_2, \cdots, y_s)$. Suppose that DMUs A and B have the same amount of inputs and outputs, ie, $\boldsymbol{x}_A = \boldsymbol{x}_B$ and $\boldsymbol{y}_A = \boldsymbol{y}_B$. Assume further that the unit price of DMU A is twice that of DMU B for each output, ie, $\boldsymbol{p}_A = 2\boldsymbol{p}_B$. Under these assumptions, we have the following theorem:

Theorem 4. Both DMUs A and B have the same price (overall) and allocative efficiencies.

Proof. Since DMUs A and B have the same inputs and outputs, they have the same technical efficiency, ie, $\varphi_A^* = \varphi_B^*$.

The revenue efficiency of DMU A (or DMU B) can be obtained by solving the following LP:

$$\begin{aligned}
\max \quad & \boldsymbol{p}_A\boldsymbol{y}(= 2\boldsymbol{p}_B\boldsymbol{y}) & \\
\text{subject to} \quad & x_{iA}(= x_{iB}) \geq \sum_{j=1}^{n} \lambda_j x_{ij}, & i = 1, \cdots, m \\
& y_r \leq \sum_{j=1}^{n} \lambda_j y_{rj}, & r = 1, \cdots, s \\
& \lambda_j \geq 0 & \forall j
\end{aligned}$$

Apparently, DMUs A and B have the same optimal solution (outputs) $\boldsymbol{y}_A^* = \boldsymbol{y}_B^*$, and hence the same revenue efficiency, since we have:

$$\alpha_A^* = \boldsymbol{p}_A\boldsymbol{y}_A/\boldsymbol{p}_A\boldsymbol{y}_A^* = 2\boldsymbol{p}_B\boldsymbol{y}_B/2\boldsymbol{p}_B\boldsymbol{y}_B^* = \boldsymbol{p}_B\boldsymbol{y}_B/\boldsymbol{p}_B\boldsymbol{y}_B^* = \alpha_B^*.$$

□

They also have the same allocative efficiency by definition 1. This also sounds very strange, since DMUs A and B have the same revenue and allocative efficiencies even though the price of DMU B is half that of DMU A.

4.3 A new scheme

The previous two sections reveal the shortcomings and irrationality of the revenue and allocative efficiencies proposed thus far.

These shortcomings are caused by the structure of the supposed production possibility set P as defined by:

$$P = \left\{(x, y) \middle| x \geq X\lambda,\ y \leq Y\lambda,\ \lambda \geq 0\right\}$$

The production possibility set P is defined only on the basis of the technical factors $X = (x_1, \cdots, x_n) \in \mathbb{R}^{m \times n}$ and $Y = (y_1, \cdots, y_n) \in \mathbb{R}^{s \times n}$ and has no concern with the prices of the outputs $P = (p_1, \cdots, p_n)$. Banihashemi and Tohidi [2] define a set of new production possibility set based on revenue as follows:

$$P_p = \{(x, \bar{y}) | x \geq X\lambda,\ \bar{y} \leq \bar{Y}\lambda,\ \lambda \geq 0\}$$

where $\bar{Y} = (\bar{y}_1, \cdots, \bar{y}_n)$ and $\bar{y}_j$ assuming that the matrices P and Y are non-negative, and all inputs are revenue-oriented. Another assumption is that the elements $\bar{y}_{ij} = (p_{ij}, y_{ij})\ \forall(i,j)$ are in homogeneous units, e.g., \$, so that the multiplication of these elements is significant. Based on the definition of the set of new possible generation P_p, the new technical efficiency $\bar{\varphi}^*$ is given as the optimal solution to the linear programming problem:

$$\begin{aligned} \bar{\varphi}^* = \max \quad & \bar{\varphi} \\ \text{subject to} \quad & x_o \geq X\lambda \\ & \bar{\varphi}\bar{y}_o \leq \bar{Y}\lambda \\ & \lambda \geq 0 \end{aligned}$$

The new revenue efficiency $\bar{\alpha}^*$ is as follows:

$$\bar{\alpha}^* = e\bar{y}_o / e\bar{y}_o^*$$

where $e \in \mathrm{R}^m$, is a row vector with the elements 1 and $\bar{y}_o^*$ is the solution to the linear programming problem below:

$$\begin{aligned} [Nrevenue] \quad \max \quad & e\bar{y} \\ \text{subject to} \quad & x_o \geq X\lambda \\ & \varphi\bar{y} \leq \bar{Y}\lambda \\ & \lambda \geq 0 \end{aligned}$$

5 Network Data Envelopment Analysis Based on SBM Model

The common DEA models which measure the relative efficiency of multiple input/ output decision-maker units may experience drawbacks such as neglecting intermediate products or linked activities. In this section, the network data envelopment analysis and the parameters of its production probability set are discussed.

Suppose n is the decision maker available in Section K. m_k and r_k are the numbers of inputs and outputs in the k^{th} section. The link from division k to division h is represented by (h,k) and the set of all links is shown by L. The observed data is $\{x_j^k \in R_+^{m_k}\}(j = 1, \cdots, n, k = 1, \cdots, K)$, $\{y_j^k \in R_+^{r_k}\}(j = 1, \cdots, n, k = 1, \cdots, K)$ and $\{z_j^{(k,h)} \in R_+^{t_{(k,h)}}\}(j = 1, \cdots, n, (k,h) \in L)$.

Thus, the production possibility set in network data envelopement analysis will be:

$$P = \Big\{(x^k, y^k, z^{(h,k)}) \big| x^k \geq X^k\lambda^k, y^k \leq Y^k\lambda^k, z^{(k,h)} = z^{(k,h)}\lambda^k\,(as\,outputs\,k), z^{(k,h)} = z^{(k,h)}\lambda^h(as\,inputs\,h), \lambda \geq 0\Big\}$$

Assume that the following model (with input nature) has a variable returns to scale and DMU_o, $(o = 1, \cdots, n)$ unit under evaluation. Since the SBM model needs to have positive data, this paper assumes that all data are positive.

$$
\begin{aligned}
[NSBM] \quad \theta_0 = \min \ & \sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k} \Big(\sum_{i=1}^{m_k} \frac{s_i^{k-}}{x_{io}^k} \Big) \right] \\
\text{subject to} \quad & x_o^k = X^k \lambda^k + s^{k-} \\
& y_o^k = Y^k \lambda^k - s^{k+} \\
& \lambda^k, \lambda^h, s^{k-}, s^{k+} \geq 0 \\
& z_o^{(k,h)} = z^{(k,h)} \lambda^k \ \big(\forall (k,h)\big), \qquad (a) \\
& z_o^{(k,h)} = z^{(k,h)} \lambda^h \ \big(\forall (k,h)\big), \\
& \text{or} \\
& z^{(k,h)} \lambda^k = z^{(k,h)} \lambda^h \ \big(\forall (k,h)\big), \qquad (b)
\end{aligned}
$$

Where z Where $z^{(k,h)} = \big(z_1^{(k,h)}, \cdots, z_n^{(k,h)}\big) \in R^{t_{(k,h)} \times n}$, $X^k = (x_1^k, \cdots, x_n^k) \in R^{m_k \times n}$, $y^k = (y_1^k, \cdots, y_n^k) \in R^{r_k \times n}$, s^{k-} (s^{k+}) are slacks vectors of the input (output). Given the link constraints, there are several choices that can be made in two possible ways:

(a) In the first case, the values of fixed intermediate current are taken into account.

$$
\begin{aligned}
& z_o^{(k,h)} = z^{(k,h)} \lambda^k \ \big(\forall (k,h)\big), \qquad (a) \\
& z_o^{(k,h)} = z^{(k,h)} \lambda^h \ \big(\forall (k,h)\big)
\end{aligned}
$$

(b)In the second case, the values of the average flow in the link can be freely reduced or increased.

$$z^{(k,h)} \lambda^k = z^{(k,h)} \lambda^h \ \big(\forall (k,h)\big), \qquad (b)$$

6 Revenue Efficiency in Network DEA

In this section we deal New Network Revenue Efficiency (NNRE) on Network Slack Based Measure (NSBM) that prices play a role in the PPS on output. The production possibility set based on price for the network data envelopment analysis is [2]:

$$
\begin{aligned}
P_p = \Big\{ & (x^k, \bar{y}^k, \bar{z}^{(k,h)}) \big| x^k \geq X^k \lambda^k, \bar{y}^k \leq \bar{Y}^k \lambda^k, \ \bar{z}^{(k,h)} = \bar{z}^{(k,h)} \lambda^k (as\, outputs\, k), z^{(k,h)} \\
& = z^{(k,h)} \lambda^h (as\, inputs\, h), \ e\lambda^k = 1, \ \lambda \geq 0 \Big\}
\end{aligned}
$$

where

$$
\begin{aligned}
& \bar{Y}^k = (\bar{y}_1^k, \cdots, \bar{y}_n^k), && \bar{y}_j^k = (p_{1j}^k y_{1j}^k, \cdots, p_{r_k j}^k y_{r_k j}^k) \\
& \bar{z}^{(k,h)} = \big(\bar{z}_1^{(k,h)}, \cdots, \bar{z}_n^{(k,h)}\big), && \bar{z}_j^{(k,h)} = \big(c_{1j}^k z_{1j}^{(k,h)}, \cdots, c_{r_k j}^k z_{r_k j}^{(k,h)}\big)
\end{aligned}
$$

Based on this set, a new production possibility, $\bar{\alpha}^{*k}$, is obtained from the following linear programming problem:

$$
\begin{aligned}
[NNRE] \quad & \max \sum_{k=1}^{K} \bar{y}^k + \sum_h \bar{z}^{(k,h)} & & \\
\text{subject to} \quad & x_o^k \geq X^k \lambda^k, & & k = 1, \cdots, K \\
& \bar{y}^k \leq \bar{Y}^k \lambda^k, & & k = 1, \cdots, K \\
& \bar{z}_o^{(k,h)} = \bar{z}^{(k,h)} \lambda^h \ \big(\forall (k,h)\big) & & (a) \\
& z_o^{(k,h)} = z^{(k,h)} \lambda^k \ \big(\forall (k,h)\big), & & \\
& \text{or} & & \\
& \bar{z}^{(k,h)} \lambda^k = \bar{z}^{(k,h)} \lambda^h \ \big(\forall (k,h)\big) & & (b) \qquad (P5) \\
& e\lambda^k = 1, & & \\
& \lambda^k, \ \lambda^h \geq 0 & &
\end{aligned}
$$

and

$$
\bar{\alpha}^{*k} = \sum_{k=1}^{K} \bar{y}_o^{*k} + \sum_h \bar{z}_o^{(k,h)} \Big/ \sum_{k=1}^{K} \bar{y}_o^{*k} + \sum_h \bar{z}_o^{*(k,h)}
$$

Where $e \in \mathbb{R}^m$, a row vector with elements, equals 1 and $\bar{y}_o^*$, $\bar{z}_o^*$ are optimal solutions for model $(P5)$.

7 Proposed Fuzzy Revenue Efficiency Method in Fully Fuzzy Network Data Analysis

In the real world, input-output data and their corresponding prices are not accurately observed and may be available in inappropriate forms such as fuzzy numbers, in particular triangular fuzzy numbers. Many researchers investigated the revenue efficiency with fuzzy and intermediate data. In these studies only, the decision parameters are considered as fuzzy and the decision variables are precise quantifiers. However, in this paper, we use full-fuzzy models of network data envelopment analysis to measure the revenue efficiency in a fully fuzzy environment in which all decision-making parameters and variables are represented by triangular fuzzy numbers.

To measure fuzzy revenue efficiency in network data envelopment analysis, we extend the model (4) to a completely fuzzy environment. Suppose that the decision maker unit is available in Section K. m_k and r_k are the number of fuzzy inputs and outputs in the k-section. The link from section k to part h is represented by (k, h) and the set of all links with L. The observed fuzzy data $j = 1, \cdots, n$, $k = 1, \cdots, K$ $\tilde{x}_j^k$, $\tilde{y}_j^k$, $\tilde{z}_j^{(k,h)}$ and $\tilde{p}_j^k$ respectively contain the input and Fuzzy outputs in each section, fuzzy link activities from section k to section h as well as the revenue of the fuzzy input units in each section. If these data are triangular fuzzy numbers, we will have:

$$\begin{aligned}
&\tilde{x}_j^k = \left(x_j^{l,k}, x_j^{m,k}, x_j^{u,k}\right), && j=1,\cdots,n, \quad k=1,\cdots,K\\
&\tilde{y}_j^k = \left(y_j^{l,k}, y_j^{m,k}, y_j^{u,k}\right), && j=1,\cdots,n, \quad k=1,\cdots,K\\
&\tilde{p}_j^k = \left(p_j^{l,k}, p_j^{m,k}, p_j^{u,k}\right), && j=1,\cdots,n, \quad k=1,\cdots,K\\
&\tilde{z}_j^{(k,h)} = \left(z_j^{l,(k,h)}, z_j^{m,(k,h)}, z_j^{u,(k,h)}\right), && j=1,\cdots,n, \quad (k,h)\in L
\end{aligned}$$

According to the above, the model ($P5$) will become a fully fuzzy model as follows:

$$\begin{aligned}
[FFNNRE] \quad & \min \sum_{k=1}^{K} \tilde{\bar{y}}^k \oplus \sum_h \tilde{\bar{z}}^{(k,h)} \\
\text{subject to} \quad & \tilde{x}_o^k \succcurlyeq \sum_{j=1}^{n} \tilde{X}_j^k \otimes \tilde{\lambda}_j^k, && k=1,\cdots,K \\
& \tilde{\bar{y}}^k \preccurlyeq \sum_{j=1}^{n} \tilde{\bar{y}}_j^k \otimes \tilde{\lambda}_j^k, && k=1,\cdots,K \qquad (P6)\\
& \tilde{\bar{z}}_o^{(k,h)} \approx \sum_{j=1}^{n} \tilde{z}_j^{(k,h)} \otimes \tilde{\lambda}_j^k, && \forall (k,h) \qquad (a)\\
& \tilde{\bar{z}}_o^{(k,h)} \approx \sum_{j=1}^{n} \tilde{z}_j^{(k,h)} \otimes \tilde{\lambda}_j^h, && \forall (k,h)\\
& \text{or} \\
& \sum_{j=1}^{n} \tilde{\bar{z}}_j^{(k,h)} \otimes \tilde{\lambda}_j^k \approx \sum_{j=1}^{n} \tilde{\bar{z}}_j^{(k,h)} \otimes \tilde{\lambda}_j^h, && \forall (k,h) \qquad (b)\\
& \sum_{j=1}^{n} \tilde{\lambda}_j^k \approx \tilde{1} \\
& \tilde{\lambda}_j^k,\ \tilde{\lambda}_j^h \succcurlyeq \tilde{0} && \forall j,k
\end{aligned}$$

The model ($P6$) is a fuzzy revenue envelopment model in the Fuzzy Network Data Envelopment Analysis. After replacing the triangular fuzzy variables and parameters in model ($P6$) and using mathematical operations on triangular fuzzy numbers and steps of the Nasseri algorithm, the full-fuzzy linear programming model ($P6$) becomes the crisp linear programming:

$$\max \ \frac{1}{4}\left[\sum_{k=1}^{K} \bar{y}^{l,k} + \sum_h \bar{z}^{l,(k,h)} + 2\Big(\sum_{k=1}^{K} \bar{y}^{m,k} + \sum_h \bar{z}^{m,(k,h)}\Big) + \sum_{k=1}^{K} \bar{y}^{u,k} + \sum_h \bar{z}^{u,(k,h)}\right]$$

$$\begin{aligned}
\text{subject to} \quad & x_o^{l,k} \geq \sum_{j=1}^{n} X_j^{l,k} \lambda_j^{l,k}, && k=1,\cdots,K\\
& x_o^{m,k} \geq \sum_{j=1}^{n} X_j^{m,k} \lambda_j^{l,k}, && k=1,\cdots,K\\
& x_o^{u,k} \geq \sum_{j=1}^{n} X_j^{u,k} \lambda_j^{l,k}, && k=1,\cdots,K
\end{aligned}$$

$$\bar{y}^{l,k} \leq \sum_{j=1}^{n} \bar{y}_j^{l,k} \lambda_j^{l,k}, \qquad k = 1, \cdots, K \qquad (P7)$$

$$\bar{y}^{m,k} \leq \sum_{j=1}^{n} \bar{y}_j^{m,k} \lambda_j^{m,k}, \qquad k = 1, \cdots, K$$

$$\bar{y}^{u,k} \leq \sum_{j=1}^{n} \bar{y}_j^{u,k} \lambda_j^{u,k}, \qquad k = 1, \cdots, K$$

$$\bar{z}_o^{l,(k,h)} = \sum_{j=1}^{n} \bar{z}_j^{l,(k,h)} \lambda_j^{l,k}, \qquad \forall (k,h)$$

$$\bar{z}_o^{m,(k,h)} = \sum_{j=1}^{n} \bar{z}_j^{m,(k,h)} \lambda_j^{m,k}, \qquad \forall (k,h)$$

$$\bar{z}_o^{u,(k,h)} = \sum_{j=1}^{n} \bar{z}_j^{u,(k,h)} \lambda_j^{u,k}, \qquad \forall (k,h)$$

$$z_o^{l,(k,h)} = \sum_{j=1}^{n} z_j^{l,(k,h)} \lambda_j^{l,h}, \qquad \forall (k,h) \qquad (a)$$

$$z_o^{m,(k,h)} = \sum_{j=1}^{n} z_j^{m,(k,h)} \lambda_j^{m,h}, \qquad \forall (k,h)$$

$$z_o^{u,(k,h)} = \sum_{j=1}^{n} z_j^{u,(k,h)} \lambda_j^{u,h}, \qquad \forall (k,h)$$

or

$$\sum_{j=1}^{n} \bar{z}_j^{l,(k,h)} \lambda_j^{l,k} = \sum_{j=1}^{n} \bar{z}_j^{l,(k,h)} \lambda_j^{l,h}, \qquad \forall (k,h)$$

$$\sum_{j=1}^{n} \bar{z}_j^{m,(k,h)} \lambda_j^{m,k} = \sum_{j=1}^{n} \bar{z}_j^{m,(k,h)} \lambda_j^{m,h}, \qquad \forall (k,h) \qquad (b)$$

$$\sum_{j=1}^{n} \bar{z}_j^{u,(k,h)} \lambda_j^{u,k} = \sum_{j=1}^{n} \bar{z}_j^{u,(k,h)} \lambda_j^{u,h}, \qquad \forall (k,h)$$

$$\sum_{j=1}^{n} \lambda_j^{l,k} = 1, \quad \sum_{j=1}^{n} \lambda_j^{m,k} = 1, \quad \sum_{j=1}^{n} \lambda_j^{u,k} = 1, \qquad k = 1, \cdots, K$$

$$\lambda_j^{l,k} \geq 0, \ \lambda_j^{m,k} - \lambda_j^{l,k} \geq 0, \ \lambda_j^{u,k} - \lambda_j^{m,k} \geq 0 \qquad \forall j,k$$

$$\bar{y}_j^{l,k} \geq 0, \ \bar{y}_j^{m,k} - \bar{y}_j^{l,k} \geq 0, \ \bar{y}_j^{u,k} - \bar{y}_j^{m,k} \geq 0 \qquad \forall j,k$$

$$\bar{z}_j^{l,k} \geq 0, \ \bar{z}_j^{m,k} - \bar{z}_j^{l,k} \geq 0, \ \bar{z}_j^{u,k} - \bar{z}_j^{m,k} \geq 0 \qquad \forall j,k$$

Theorem 5. Model $(P7)$ is a feasible model.

Proof. This model has a feasible solution as follows :

$$\begin{array}{lll}
\lambda_o^{l,k}=1, & \lambda_j^{l,k}=0, & j\neq o\\
\lambda_o^{m,k}=1, & \lambda_j^{m,k}=0, & j\neq o\\
\lambda_o^{u,k}=1, & \lambda_j^{u,k}=0, & j\neq o\\
\lambda_o^{l,h}=1, & \lambda_j^{l,h}=0, & j\neq o\\
\lambda_o^{m,h}=1, & \lambda_j^{m,h}=0, & j\neq o\\
\lambda_o^{u,h}=1, & \lambda_j^{u,h}=0, & j\neq o\\
\bar{y}^{l,k}=\bar{y}_o^{l,k} & \bar{y}^{m,k}=\bar{y}_o^{m,k} & \bar{y}^{u,k}=\bar{y}_o^{u,k}
\end{array}$$

And with considering (b)

$$\bar{z}_o^{l,(k,h)}=\bar{z}_o^{l,(k,h)} \qquad \bar{z}_o^{m,(k,h)}=\bar{z}_o^{m,(k,h)}, \qquad \bar{z}_o^{u,(k,h)}=\bar{z}_o^{u,(k,h)}$$

□

Theorem 6. The optimal solution for the model (P7) will be a model optimization solution (P6). The proof of this is similar to the proof of Theorem 1.

Definition 13. The fuzzy cost efficiency of the i^{th} DMU in the FFDEA is defined as the ratio of the minimum fuzzy cost to the observed fuzzy cost of DMU_i:

$$\begin{aligned}
\tilde{\alpha}_i^{*k} &= \frac{\sum_{k=1}^{K}\tilde{y}_i^{*k}\oplus\sum_h\tilde{z}_i^{*(k,h)}}{\sum_{k=1}^{K}\tilde{x}_i^{k}\oplus\sum_h\tilde{z}_i^{*(k,h)}}\\
&= \frac{\left(\sum_{k=1}^{K}\bar{y}_i^{l,k}+\sum_h\bar{z}_i^{l,(k,h)},\sum_{k=1}^{K}\bar{y}_i^{m,k}+\sum_h\bar{z}_i^{m,(k,h)},\sum_{k=1}^{K}\bar{y}_i^{u,k}+\sum_h\bar{z}_i^{u,(k,h)}\right)}{\left(\sum_{k=1}^{K}\bar{y}_i^{l,k*}+\sum_h\bar{z}_i^{l,(k,h)*},\sum_{k=1}^{K}\bar{y}_i^{m,k*}+\sum_h\bar{z}_i^{m,(k,h)*},\sum_{k=1}^{K}\bar{y}_i^{u,k*}+\sum_h\bar{z}_i^{u,(k,h)*}\right)}\\
&= \left(\frac{\sum_{k=1}^{K}\bar{y}_i^{l,k}+\sum_h\bar{z}_i^{l,(k,h)},\sum_{k=1}^{K}\bar{y}_i^{m,k}+\sum_h\bar{z}_i^{m,(k,h)},\sum_{k=1}^{K}\bar{y}_i^{u,k}+\sum_h\bar{z}_i^{u,(k,h)}}{\sum_{k=1}^{K}\bar{y}_i^{u,k*}+\sum_h\bar{z}_i^{u,(k,h)*},\sum_{k=1}^{K}\bar{y}_i^{m,k*}+\sum_h\bar{z}_i^{m,(k,h)*},\sum_{k=1}^{K}\bar{y}_i^{l,k*}+\sum_h\bar{z}_i^{l,(k,h)*}}\right)
\end{aligned}$$

where $(\bar{y}_i^{l,k*},\bar{y}_i^{m,k*},\bar{y}_i^{u,k*} \quad \forall i,k,h)$ $(\bar{z}_i^{l,(k,h)*},\bar{z}_i^{m,(k,h)*},\bar{z}_i^{u,(k,h)*})$ are the optimal solutions obtained from model (p6).

Definition 14. i^{th} DMU in the network data envelopment analysis is called Fuzzy Cost Efficiency if the observed Fuzzy Cost and the minimum Fuzzy Cost equal DMU_i, that is,

$$\sum_{k=1}^{K}\tilde{y}_i^{k}\oplus\sum_h\tilde{z}_i^{(k,h)}\approx\sum_{k=1}^{K}\tilde{y}_i^{*}\oplus\sum_h\tilde{z}_i^{*(k,h)}$$

$$\Re\Big(\sum_{k=1}^{K}\tilde{y}_i^{k}\oplus\sum_h\tilde{z}_i^{(k,h)}\Big)\approx\Re\Big(\sum_{k=1}^{K}\tilde{y}_i^{*}\oplus\sum_h\tilde{z}_i^{*(k,h)}\Big)$$

8 Numerical example

In this section, an illustrative example of electric power companies are presented for describing network DEA. As we know, the vertically integrated electric power companies consist of several divisions such as generation, transmission and distribution. For illustrative purpose, ten

vertically integrated electric power companies in the U.S in 1994 [16]. The inputs, outputs and links are as follows:
Generation (Div1):
Input1 = Labor input (number of employees)
Transmission (Div2):
Input2 = Labor input (number of employees)
Output2 = Electric power sold to large customers
Distribution (Div3):
Input3 = Labor input (number of employees)
Output3 = Electric power sold to small customers
Link (1-2) = Electric power generated (output from Generation Devision and input to Transmission Devision)
Link (2-3) = Electric power sent (output from Transmission Devision and input to Distribution Devision) Here, it is assumed that the intermediate flow rates are able to rise or fall freely in the link, so that the proposed model for evaluating the fuzzy revenue efficiency will be as follows:

$$\max \ \frac{1}{4}\left[\sum_{k=1}^{K}\bar{y}^{l,k}+\sum_{h}\bar{z}^{l,(k,h)}+2\Big(\sum_{k=1}^{K}\bar{y}^{m,k}+\sum_{h}\bar{z}^{m,(k,h)}\Big)+\sum_{k=1}^{K}\bar{y}^{u,k}+\sum_{h}\bar{z}^{u,(k,h)}\right]$$

$$\begin{aligned}
\text{subject to}\quad & x_o^{l,k} \geq \sum_{j=1}^{n} X_j^{l,k}\lambda_j^{l,k}, && k=1,\cdots,K\\
& x_o^{m,k} \geq \sum_{j=1}^{n} X_j^{m,k}\lambda_j^{m,k}, && k=1,\cdots,K\\
& x_o^{u,k} \geq \sum_{j=1}^{n} X_j^{u,k}\lambda_j^{u,k}, && k=1,\cdots,K\\
& \bar{y}^{l,k} \leq \sum_{j=1}^{n} \bar{y}_j^{l,k}\lambda_j^{l,k}, && k=1,\cdots,K\\
& \bar{y}^{m,k} \leq \sum_{j=1}^{n} \bar{y}_j^{m,k}\lambda_j^{m,k}, && k=1,\cdots,K\\
& \bar{y}^{u,k} \leq \sum_{j=1}^{n} \bar{y}_j^{u,k}\lambda_j^{u,k}, && k=1,\cdots,K\\
& \sum_{j=1}^{n} \bar{z}_j^{l,(k,h)}\lambda_j^{l,k} = \sum_{j=1}^{n} z_j^{l,(k,h)}\lambda_j^{l,h}, && \forall (k,h)\\
& \sum_{j=1}^{n} \bar{z}_j^{m,(k,h)}\lambda_j^{m,k} = \sum_{j=1}^{n} z_j^{m,(k,h)}\lambda_j^{m,h}, && \forall (k,h)\\
& \sum_{j=1}^{n} \bar{z}_j^{u,(k,h)}\lambda_j^{u,k} = \sum_{j=1}^{n} z_j^{u,(k,h)}\lambda_j^{u,h}, && \forall (k,h)\\
& \lambda_j^{l,k} \geq 0, \quad \lambda_j^{m,k}-\lambda_j^{l,k} \geq 0, \quad \lambda_j^{u,k}-\lambda_j^{m,k} \geq 0 && \forall j,k\\
& \bar{y}_j^{l,k} \geq 0, \quad \bar{y}_j^{m,k}-\bar{y}_j^{l,k} \geq 0, \quad \bar{y}_j^{u,k}-\bar{y}_j^{m,k} \geq 0, && \forall j,k
\end{aligned}$$

$$\bar{z}_j^{l,(k,h)} \geq 0, \quad \bar{z}_j^{m,(k,h)} - \bar{z}_j^{l,(k,h)} \geq 0, \quad \bar{z}_j^{u,(k,h)} - \bar{z}_j^{m,(k,h)} \geq 0, \quad \forall j, k, h$$

Table 1 contains the fuzzy inputs, fuzzy outputs, and fuzzy revenues of each division.

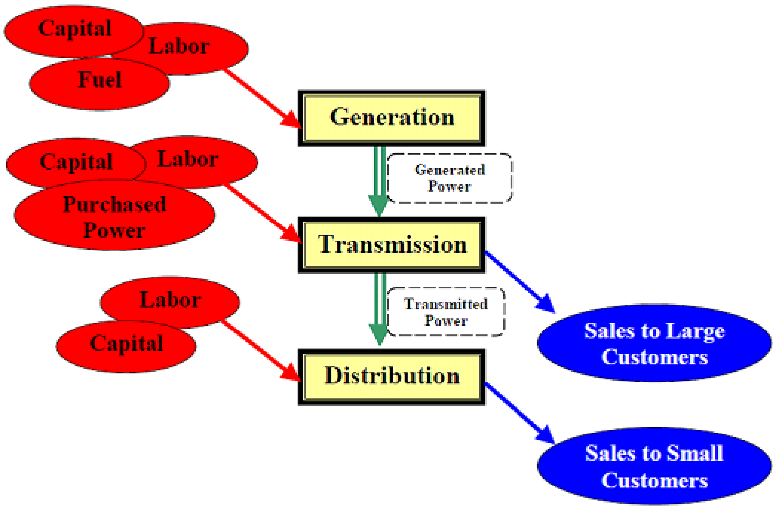

Figure 2: Vertically integrated electric power companies

The revenue of the input and output links is also given in Table 2.

Table 1: Fuzzy inputs, fuzzy outputs, fuzzy input cost in three divisions

	Div1	Div2		Div3			
DMU	Input1	Input2	Output3	P2	Input3	Output3	P3
A	(0.836,0.838,0.840)	(0.275,0.277,0.279)	(0.876,0.879,0.881)	(896,900,903)	(0.960,0.962,0.965)	(0.335,0.337,0.340)	(685,687,689)
B	(1.231,1.233,1.235)	(0.130,0.132,0.133)	(0.535,0.538,0.540)	(737,739,742)	(0.440,0.443,0.445)	(0.15,0.18,0.20)	(190,194,196)
C	(0.318,0.321,0.323)	(0.042,0.045,0.048)	(0.909,0.911,0.914)	(138,142,145)	(0.482,0.485,0.487)	(0.195,0.198,0.200)	(280,285,287)
D	(1.480.1.483,1.485)	(0.110,0.111,0.113)	(0.55,0.57,,0.59)	(860,863,865)	(0.465,0,467,0.470)	(0.488,0.491,0.495)	(398,401,404)
E	(1.590,1.592,1.595)	(0.205,0.208,0.211)	(1.085,1.086,1.089)	(305,307,310)	(1.070,1.073,1.075)	(0.370,0.372,0.375)	(175,179,182)
F	(0.76,0.79,0.81)	(0.136,0.139,0.141)	(0.720,0.722,0.724)	(1198,1200,1203)	(0.543,0.545,0.548)	(0.250,0.253,0.255)	(1052,1054,1056)
G	(0.449,0.451,0.454)	(0.073,0.075,0.077)	(0.507,0.509,0.511)	(268,270,273)	(0.365,0.366,0.368)	(0.238,0.241,0.244)	(390,394,396)
H	(0.405,0.408,0.410)	(0.072,0.074,0.076)	(0.617,0.619,0.621)	(985,987,990)	(0.226,0.229,0.231)	(0.095,0.097,0.099)	(272,276,280)
I	(1.860,1.864,1.865)	(0.059,0.061,0.063)	(1.021,1.023,1.025)	(354,356,358)	(0.689,0.691,0.693)	(0.35,0.38,0.40)	(838,840,843)
J	(1.220,1.222,1.225)	(0.147,0.149,0.151)	(0.765,0.769,0.771)	(467,470,472)	(0.336,0.337,0.339)	(0.175,0.178,0.180)	(159,161,164)

The above model is solved using GAMS software and the results are shown in Table 3.

As Table 3 shows none of the decision making units are revenue efficiency. Indeed, one of the major drawbacks of the network models is that the full efficiency cannot be achieved in most of the cases. To solve this issue, efficiency of each unit can be devided to the maximum efficiency, resulting to deriving the relative efficiency (Table 3, column 4). In this case, unit H is the relative revenue efficiency and units A, C, D, F and G have the relative revenue efficiency more than half.

Table 2: Fuzzy unit input link revenue

Link			
Link12	Lp1	Link23	Lp2
(0.891,0.894,0.897)	(945,947,950)	(0.360,0.362,0.365)	(1031,1034,1036)
(0.675,0.678,0.780)	(680,682,685)	(0.185,0.188,0.190)	(986,989,992)
(0.835,0.836,0.838)	(700,705,708)	(0.205,0.207,0.210)	(750,752,755)
(0.865,0.869,0.872)	(1125,1128,1130)	(0.514,0.516,0.520)	(1109,1111,1113)
(0.690,0.693,0.695)	(490,492,495)	(0.405,0.407,0.410)	(850,852,855)
(0.961,0.966,0.970)	(665,670,673)	(0.265,0.269,0.273)	(640,642,645)
(0.645,0.647,0.650)	(1085,1087,1090)	(0.255,0.257,0.259)	(820,824,826)
(0.752,0.756,0.760)	(924,926,930)	(0.101,0.103,0.105)	(970,973,975)
(1.189,1.191,1.194)	(630,634,638)	(0.400,0.402,0.405)	(910,913,915)
(0.790,0.792,0.795)	(775,779,782)	(0.185,0.187,0.190)	(645,647,650)

Table 3: Evaluating and ranking revenue efficiency

$DMUs$	$\tilde{\tilde{\alpha}}^{*k}$	$R(\tilde{\tilde{\alpha}}^{*k})$	Relative Efficiency	Rank
A	(0.433,.0.648,0.734)	0.648	0.733	4
B	(0.130,0.331,0.450)	0.331	0.374	8
C	(0.435,0.680,0.872)	0.680	0.769	3
D	(0.435,0.553,0.754)	0.553	0.625	6
E	(0.125,0.263,0.365)	0.263	0.297	10
F	(0.534,0.709,0.845)	0.709	0.802	2
G	(0.456,0.647,0.745)	0.647	0.732	5
H	(0.534,0.884,0.915)	0.884	1	1
I	(0.234,0.403,0.478)	0.403	0.456	7
J	(0.25,0.33,0.56)	0.33	0.373	9

9 Conclusion

Given the importance of revenue efficiency in the management and economic sectors as well as inaccuracies in real-world data, this paper proposes a new idea of the extension of classical NNRE model to fully fuzzy environments for dealing with the practical situations more realistically. A FFNNRE model has been developed where input–output data and their corresponding prices are taken in triangular membership forms. A method based on ranking function approach is presented to transform FFNNRE model into the crisp linear programming problem. The final FFNNRE measures are then defined as TFNs. Finally, using the presented ranking function in the article, the DMUs are ranked based on revenue efficiency.

Since revenue efficiency sensitivity analysis helps the manager or decision maker to modify the amount of outputs under evaluation to maximize revenue . Therefore, future work can include sensitivity analysis of performance, as well as finding the appropriate stability area to maintain revenue efficiency in precise and imprecise network data envelopment analysis.

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Persian Abstracts

توابع شبه‌شکاف و شکاف برای مسائل نیمه-نامتناهی چندهدفه غیرهموار

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تاریخ دریافت: ۱۸ دی ۱۳۹۸ تاریخ پذیرش: ۱۴ فروردین ۱۳۹۹

چکیده

در این مقاله ما به معرفی و مطالعه چند تابع شکاف تک‌مقداری جدید برای مسائل بهینه‌سازی چندهدفه نیمه نامتناهی غیرمشتق‌پذیر با داده‌های موضعا لیپ‌شیتز پرداخته‌ایم. از آنجا که یکی از خواص اصلی هر تابع شکافی برای یک مسئله بهینه‌سازی، توانایی آن در مشخص‌سازی جواب‌های آن مسئله است، این خاصیت توابع شکاف جدید معرفی شده نیز ارائه شده است. تمامی احکام بر حسب زیرمشتق کلارک بیان شده‌اند.

کلمات کلیدی

بهینه‌سازی چندهدفه، برنامه‌ریزی نیمه-نامتناهی، تابع شکاف، زیرمشتق کلارک.

یک تابع شکاف حقیقی مقدار برای مسائل برنامه ریزی چند هدفی نیمه نامتناهی غیر هموار

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تاریخ دریافت: ۲۵ فروردین ۱۳۹۸ تاریخ پذیرش: ۲۳ مهر ۱۳۹۸

چکیده

ما در این مقاله برای یک مسئله برنامه ریزی چند هدفه غیر هموار که توسط تعداد بینهایت قید تعریف می‌شود تابع شکاف جدیدی را معرفی می‌کنیم که تعمیم این مفهوم در مقالات دیگر است. آنگاه ما کارایی، کارایی ضعیف و کارایی سره مسئله فوق را توسط این تابع شکاف جدید مشخص سازی می‌کنیم تمام مفاهیم ما بر مبنای مفهوم توابع $\Phi, \rho-$ اینوکس و زیر مشتق کلارک تنظیم گشته‌اند.

کلمات کلیدی

برنامه ریزی نیمه نامتناهی، بهینه سازی چندهدفه، کیفیت محدود، شرایط بهینگی، تابع شکاف.

انتخاب نقاط مرکزی توابع پایه ای شعاعی با کمک تکنیک پرامیتی

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تاریخ دریافت: ۲۵ فروردین ۱۳۹۸ تاریخ پذیرش: ۲۳ مهر ۱۳۹۸

چکیده

در این مقاله تلاش می شود که بهترین نقاط مرکزی توابع پایه شعاعی را با استفاده از تکنیک‌های تصمیم‌گیری چند معیاره (MCDM) انتخاب کنیم. دو روش مبتنی بر توابع پایه‌ای شعاعی برای حل معادلات دیفرانسیل با مشتقات جزئی مورد استفاده قرار می‌گیرد. روش اول مبتنی بر روش کانسا و روش دوم مبتنی بر درون‌یابی هرمیتی می‌باشند. علاوه بر این، با انتخاب پنج مجموعه از نقاط مرکزی: کارتزین، هم‌فاصله، چبیشف، لژاندر و لژاندر گاوس لوباتو به عنوان گزینه‌های تحقیق و متغیرهای: خطا، عدد حالت ماتریس درون‌یاب و زمان اجرا به عنوان معیارهای تاثیرگذار، گزینه‌ها با کمک تکنیک پرامیتی رتبه‌بندی گردیدند. در نهایت بهترین نقاط مرکزی بر اساس رتبه بدست آمده انتخاب گردید. این رتبه‌بندی نشان می‌دهد که روش درون‌یابی هرمیتی با استفاده از نقاط غیر یکنواخت به عنوان نقاط مرکزی مناسب‌تر از روش کانسا با هر نقطه مرکزی است.

کلمات کلیدی

تصمیم گیری چندمعیاره، توابع مرکزي شعاعي، پراميتي، درون يابي هرميت، انتخاب بهینه .

مشخص‌سازی جواب‌های موثر سره برای مسائل چندهدفه‌ی محدب با قیود غیرمشتق‌پذیر پوچ‌شونده

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تاریخ دریافت: ۲۷ آبان ۱۳۹۷ تاریخ پذیرش: ۲۳ شهریور ۱۳۹۸

چکیده

ما در این مقاله یک مسئله‌ی بهینه‌سازی چندهدفه‌ی محدب را در نظر می‌گیریم که توسط قیدهای پوچ‌شونده تعریف می‌شود. در ابتدا، یک قید تعریفی جدید برای مسئله معرفی کرده و توسط مخروط نرمال مردخویچ، یک شرط لازم برای جواب‌های موثر سره‌ی مسئله ارائه خواهیم داد. آنگاه ثابت خواهیم کرد که شرط لازم بیان شده، شرط کافی نیز برای جواب‌های موثر سره می‌باشد. احکام ما بر حسب زیرمشتق محدب فرمول‌بندی شده‌اند.

کلمات کلیدی

بهینه‌سازی چندهدفه، قیود چندهدفه، بهینه‌سازی محدب، قیدهای تعریفی.

دو روش تنظیم پارامتر دای-لیاو بر اساس معادلات سکانت اصلاح شده

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چکیده

در ادامه کار تنظیم پارامتر دای-لیاو در روش‌های گرادیان مزدوج، دو پارامتر جدید براساس معادلات سکانت اصلاح‌شده معرفی شده توسط لی و فوکوشیما، با دو رویکرد متفاوت که از یک شرط مزدوجی جدید استفاده می‌کند، ارایه کرده‌ایم. اولین پارامتر براساس روش ارایه شده توسط ژنگ و همکارانش به عنوان یک روش گرادیان مزدوج هستینس-استیفل است. دومین پارامتر براساس رویکرد شبه نیوتن است. همگرایی سراسری روش‌های پیشنهادی برای توابع محدب یکنواخت و توابع عمومی ثابت شده است. نتایج عددی با استفاده از مجموعه‌ای از مسایل CUTEr و مقایسه روش‌های پیشنهادی با تعدادی از روش‌های مشهور، به دست آمده است.

کلمات کلیدی

بهینه‌سازی نامقید، معادلات مرزی اصلاح‌شده، روش گرادیان مزدوج دای-لیاو.

یک روش تمام فازی تحلیل پوششی داده‌های شبکه‌ای برای ارزیابی کارایی درآمد براساس توابع رتبه‌بندی

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تاریخ دریافت: ۲۵ اردیبهشت ۱۳۹۸ تاریخ پذیرش: ۹ آذر ۱۳۹۸

چکیده

هدف این مقاله، ارزیابی کارایی درآمد در تحلیل پوششی داده‌های شبکه‌ای تمام فازی می‌باشد. اندازه‌گیری دقیق داده‌ها در دنیای واقعی عملا امکان‌پذیر نمی‌باشد، بنابراین فرض دقیق بودن داده‌ها در حل مسائل، فرض درستی نمی‌باشد. یکی از راه‌های مواجهه با داده‌های نادقیق، داده‌های فازی می‌باشد. در این مقاله از توابع رتبه‌بندی خطی، برای تبدیل مدل تمام فازی کارایی درآمد به یک مسئله برنامه‌ریزی خطی دقیق استفاده می‌شود و با فرض اعداد فازی مثلثی، کارایی درآمد فازی تصمیم‌گیرنده‌ها اندازه‌گیری می‌شود. در پایان، یک مثال عددی روش پیشنهادی را نشان می‌دهد.

کلمات کلیدی

تحلیل پوششی داده‌های شبکه‌ای، کارایی درآمد، برنامه‌ریزی خطی تمام فازی، تابع رتبه‌بندی.

فرم اشتراک

علاقه‌مندان به اشتراک نشریه

Control and Optimization in Applied Mathematics-COAM

می توانند فرم ذیل را تکمیل نمایند و به همراه فیش بانکی به مبلغ ۵۰۰۰۰ ریال (پنجاه هزار ریال) به شماره حساب ۲۱۷۸۶۰۹۰۰۱۰۰۷: و معادل شبای متمرکز IR ۴۲ ۰۱۷۰ ۰۰۰۰ ۰۲۱۷ ۸۶۰۹ ۰۷ ، نزد بانک ملی ایران شعبه بنفشه، کد: ۱۵۰۸ به دبیرخانه مجله ارسال دارند تا مجله برای آنان فرستاده شود.

نام:
نام موسسه یا مرکز:
آخرین مدرک تحصیلی:

نام خانوادگی:
رشته تحصیلی:
شماره اشتراک:

اشتراک از شماره............. تا شماره .... و تعداد مورد نیاز از هر شماره .... نسخه
نشانی کامل پستی:

کدپستی:
تلفن تماس:
دورنگار:
تاریخ:

تلفن همراه:
پست الکترونیکی:
امضاء: